

GROMOV-WITTEN THEORY OF TOROIDAL ORBIFOLDS AND GIT WALL-CROSSING

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ABSTRACT. Toroidal 3-orbifolds $(S^1)^6/G$, for G a finite group, were some of the earliest examples of Calabi-Yau 3-orbifolds to be studied in string theory. While much mathematical progress towards the predictions of string theory has been made in the meantime, most of it has dealt with hypersurfaces in toric varieties. As a result, very little is known about curve-counting theories on toroidal orbifolds. In this paper, we initiate a program to study mirror symmetry and the Landau-Ginzburg/Calabi-Yau (LG/CY) correspondence for toroidal orbifolds. We focus on the simplest example $[E^3/\mu_3]$, where $E \subseteq \mathbb{P}^2$ is the elliptic curve $\mathbb{V}(x_0^3 + x_1^3 + x_2^3)$. We study this orbifold from the point of GIT wall-crossing using the gauged linear sigma model, a collection of moduli spaces generalizing spaces of stable maps. Our main result is a mirror symmetry theorem that applies simultaneously to the different GIT chambers. Using this, we analyze wall-crossing behavior to obtain an LG/CY correspondence relating the genus-zero Gromov-Witten invariants of $[E^3/\mu_3]$ to generalized Fan-Jarvis-Ruan-Witten invariants.

1. INTRODUCTION

1.1. The gauged linear sigma model. Landau-Ginzburg/Calabi-Yau (LG/CY) correspondences are conjectural relations between invariants of certain moduli spaces. On one hand, Gromov-Witten theory provides a collection of “virtual curve counts” on a Calabi-Yau orbifold Z . These are integrals over the moduli stacks $\overline{\mathcal{M}}_{g,n}(Z, \beta)$ of twisted stable maps ([1]). On the other hand, Fan-Jarvis-Ruan ([20], based on ideas of Witten) constructed moduli stacks $\mathcal{W}_{g,n}^Z$ parametrizing roots of line bundles on orbifold curves. These are “combinatorial” in nature, whereas the spaces $\overline{\mathcal{M}}_{g,n}(Z, \beta)$ are “geometric”. One may generate *Fan-Jarvis-Ruan-Witten (FJRW) invariants* by integrating cohomology classes over $\mathcal{W}_{g,n}^Z$. Using motivation from string theory, Witten [35] predicted that either of these sets of invariants — Gromov-Witten or FJRW — could be computed from the other. A far-reaching conjecture was precisely formulated by Ruan ([30]), and was recently proven by Chiodo-Iritani-Ruan ([8]) in the case where Z is a Calabi-Yau hypersurface in weighted projective space. The form of the conjecture is described in more detail below.

Toroidal 3-orbifolds $[(S^1)^6/G]$ are some of the earliest examples studied in string theory ([18]). They form a rich class of very explicit Calabi-Yau orbifolds (see the classification [22]). Nevertheless, in many ways we know very little about them; for example, the program of *mirror symmetry* for Calabi-Yau orbifolds has been worked out only for complete intersections in toric stacks, whereas most toroidal orbifolds are not of this type. Similarly LG/CY correspondences have not been studied in this context.

Our goal in this paper is to initiate a program towards filling both of these gaps. Our strategy is based on a common generalization of the moduli stacks $\overline{\mathcal{M}}_{g,n}(Z, \beta)$ and $\mathcal{W}_{g,n}^Z$ above, collectively called the *gauged linear sigma model*, or GLSM. It was proposed by Witten and formulated mathematically by Fan-Jarvis-Ruan ([21]). The main feature of these new stacks is that they take as input a GIT presentation $[V //_{\theta} G]$. Dolgachev-Hu ([19]) and Thaddeus ([32]) studied how such GIT quotients change if θ crosses a wall of a certain finite chamber decomposition, and similarly the GLSM stacks depend on a chamber of this decomposition. There is a *geometric chamber* of

this decomposition whose GLSM moduli stack is $\overline{\mathcal{M}}_{g,n}(Z, \beta)$, and a so-called *pure Landau-Ginzburg (LG) chamber* whose GLSM moduli stack is $\mathcal{W}_{g,n}^Z$. The LG/CY correspondence is thus recast as a “GIT wall-crossing” phenomenon.

We fully work out the genus-zero LG/CY correspondence in the simplest example of a toroidal orbifold, $[E^3/\mu_3]$ for E an elliptic curve, and will apply our technique more generally in a subsequent article. $[E^3/\mu_3]$ is a complete intersection in a GIT quotient $[\mathbb{C}^{13} //_{\theta} (\mathbb{C}^*)^4]$. This quotient has a chamber decomposition with 16 chambers, namely the 16 hyperoctants (\pm, \pm, \pm, \pm) in \mathbb{R}^4 . We find the geometric chamber to be the hyperoctant $(+, +, +, +)$, and the pure LG-chamber to be $(-, -, -, +)$. We choose a sequence of wall-crossings

$$(+, +, +, +) \rightarrow (+, +, -, +) \rightarrow (+, -, -, +) \rightarrow (-, -, -, +)$$

connecting these two chambers, and in each of the four chambers we describe the corresponding GLSM moduli stacks as parametrizing sections of certain line bundles on orbifold curves. To each of these chambers is then associated a collection of numerical invariants.

GLSM moduli stacks depend upon an additional parameter $\epsilon \in \mathbb{Q}_{>0}$. The special cases $\overline{\mathcal{M}}_{g,n}(Z, \beta)$ and $\mathcal{W}_{g,n}^Z$ above correspond to $\epsilon \rightarrow \infty$. We use techniques based on those of Ciocan-Fontanine and Kim ([13]) to prove an all-chamber *mirror theorem*¹:

Theorem 8.1. Let $\{J^{\epsilon, \theta}\}$ be the generating functions of GLSM invariants defined in Section 7.3. Then there is an explicit invertible transformation identifying $J^{\epsilon, \theta}$ with $J^{\infty, \theta}$.

Theorem 8.1 is the core of the paper, as well as its most notable aspect. Finding an appropriate statement of mirror symmetry for each GIT chamber is the most difficult part of the LG/CY correspondence. Previous examples of LG/CY correspondences have all relied on proving mirror theorems in each chamber separately, whereas our proof is *uniform in θ* , i.e. applies simultaneously in all chambers.

Setting $\epsilon \rightarrow 0$ and $\theta = (+, +, +, +)$ recovers an instance of the *mirror theorem for toric stacks* ([16, 7]), and setting $\epsilon \rightarrow 0$ and $\theta = (-, -, -, +)$ gives a *Landau-Ginzburg mirror theorem* similar to the one in [10]. The purpose of Theorem 8.1 is that one may compute (a restriction of) $J^{0+, \theta}$ for each θ :

Corollary (Section 9). There are explicit hypergeometric functions I^{θ} that encode the GLSM invariants of $[\mathbb{C}^{13} //_{\theta} (\mathbb{C}^*)^4]$. For example (using notation defined throughout the paper), the series

$$I^{(+, +, -, +)}(q, \hbar) = \sum_{\substack{\beta_x \in \mathbb{Z}_{\geq 0} \\ \beta_y \in \mathbb{Z}_{\geq 0} \\ \beta_z \in (-1/3)\mathbb{Z}_{>0} \setminus \mathbb{Z} \\ \beta_a \in (1/3)\mathbb{Z}_{\geq 0}}} q_x^{\beta_x} q_y^{\beta_y} q_z^{\beta_z + 1/3} q_a^{\beta_a} \frac{\prod_{\substack{\rho \in \mathbf{R} \\ \beta_{\rho} < -1}} \prod_{\lfloor \beta_{\rho} \rfloor + 1 \leq \nu \leq -1} (D_{\rho} + (\beta_{\rho} - \nu)\hbar)}{\prod_{\substack{\rho \in \mathbf{R} \\ \beta_{\rho} \geq 0}} \prod_{0 \leq \nu \leq \lfloor \beta_{\rho} \rfloor - 1} (D_{\rho} + (\beta_{\rho} - \nu)\hbar)} A_x A_y 1_{\langle -\beta \rangle}$$

encodes the GLSM invariants of the chamber $(+, +, -, +)$.

Finally, we relate the functions I^{θ} across the four GIT chambers above. We think of I^{θ} as a holomorphic map from a certain space \mathcal{N} to a subquotient $\mathcal{H}(\theta)$ of the Chen-Ruan cohomology $H_{CR}^*([\mathbb{C}^{13} //_{\theta} (\mathbb{C}^*)^4])$. There are two problems with comparing the functions I^{θ} . First, they do not have the same codomain. Second, they are not defined on all of \mathcal{N} ; in fact, each is defined on a different small open set. Thus we relate them in two steps:

- (1) We find a natural sequence of graded isomorphisms (Theorem 5.4)

$$\mathcal{H}(+, +, +, +) \cong \mathcal{H}(+, +, -, +) \cong \mathcal{H}(+, -, -, +) \cong \mathcal{H}(-, -, -, +).$$

¹In fact, as ϵ moves within a chamber decomposition of $\mathbb{Q}_{>0}$, this theorem is also a type of *wall-crossing*. To avoid confusion, we use the term only to refer to walls of GIT chambers.

(2) We find a sequence of analytic continuations of I^θ on \mathcal{N} (Section 10).

The method of analytic continuation is based on [10]. Together, these prove:

Theorem 10.7. After analytic continuation and identification of GLSM state spaces, the functions I^θ differ by (explicit) linear transformations.

The first genus-zero LG/CY correspondence was proved for the quintic threefold by Chiodo and Ruan ([10]). It has since been proven for several other classes of targets, including Calabi-Yau hypersurfaces in weighted projective spaces ([8]), many classes of Calabi-Yau complete intersections in weighted projective spaces ([14, 15]), and some other examples ([28, 31]). (Acosta ([3]) also developed a similar correspondence for non-Calabi-Yau hypersurfaces in weighted projective spaces.) All of these used techniques quite different from those presented here; [8], [14], [28], and [31] used a direct computational method, and [15] (following previous work for hypersurfaces in [26]) used a reduction to the crepant transformation conjecture, previously established in the relevant cases in [17].

While toroidal 3-orbifolds are our primary motivation, we expect Theorem 8.1 to apply much more broadly, to large classes of complete intersections in GIT quotients carrying certain torus actions. Because of this, we view it as a general conceptual approach to LG/CY correspondences. In the future we hope to explore the generality in which this technique applies.

Plan of the paper. Section 2 contains background facts about GIT quotients and orbifold curves. In Sections 3 and 4, we introduce the “target” stacks $Z(\theta) \subseteq [\mathbb{C}^{13} //_\theta (\mathbb{C}^*)^4]$ and their associated moduli stacks $\text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)$. In Section 5 we define the GLSM state spaces $\mathcal{H}(\theta)$, and the GLSM invariants, which are integrals over the moduli stacks $\text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)$. In Section 6 we define natural group actions on our moduli stacks, to be used for fixed-point localization. Section 7 contains the definitions of various generating functions, which we use to prove our all-chamber mirror theorem in Section 8. Finally, in Sections 9 and 10 we compute the series $I^\theta(q, \hbar)$ for general θ , and relate the resulting formulas to each other. Section 11 is a table of notation.

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2. BACKGROUND

We work over \mathbb{C} . We denote by μ_d the group of d th roots of unity in \mathbb{C} .

2.1. Geometric invariant theory and stack quotients.

Definition 2.1. Let V be a smooth affine variety, let G be a reductive algebraic group acting on V , and let $\theta : G \rightarrow \mathbb{C}^*$ be a character of G . This defines a G -action on the trivial bundle $V \times \mathbb{C}$ by $g \cdot (v, z) = (g \cdot v, \theta(g)z)$. The G -invariant sections of $V \times \mathbb{C}$ are called θ -equivariant functions. The θ -unstable locus $V^{uns}(\theta)$ in V is the subvariety defined by the vanishing of all $N\theta$ -equivariant functions, for $N \geq 1$. The θ -semistable locus $V^{ss}(\theta)$ is the complement of $V^{uns}(\theta)$. The θ -stable locus $V^s(\theta)$ is the set of semistable points with finite G -stabilizer whose G -orbit is closed in $V^{ss}(\theta)$.

In this paper we will always have $V^s(\theta) = V^{ss}(\theta)$. In this case the GIT stack quotient $[V //_\theta G] := [V^{ss}(\theta)/G]$ is a smooth separated Deligne-Mumford stack.

Proposition 2.2. Let G be a group acting (on the left) on a variety V , and let S be any scheme. There is a natural correspondence between

- (1) Maps $f : S \rightarrow [V/G]$,
- (2) Principal G -bundles \mathcal{P} on S together with a G -equivariant map $\phi : \mathcal{P} \rightarrow V$, and

(3) *Principal G -bundles \mathcal{P} on S together with a section σ_ϕ of the associated fiber bundle $\mathcal{P} \times_G V \rightarrow S$.*

Proof. The equivalence of (1) and (2) is by definition of a map $S \rightarrow [V/G]$. We show that (2) and (3) are equivalent. A section $\sigma : S \rightarrow \mathcal{P} \times_G V$ gives a map $S \rightarrow [V/G]$ by composition with the projection $\mathcal{P} \times_G V \rightarrow [V/G]$. Conversely, given a G -equivariant map $\mathcal{P} \rightarrow V$, we define a section $S \rightarrow \mathcal{P} \times_G V$ by mapping $s \mapsto (p, \phi(p))$. It is straightforward to check these are inverse to each other. \square

With V , G , and S as in Proposition 2.2, let $\rho : G \rightarrow \mathbb{C}^*$ be a character of G . This induces a G -equivariant structure on the trivial bundle $V \times \mathbb{C}$ by

$$g \cdot (v, z) := (g \cdot v, \rho(g)z).$$

(We write $V \times \mathbb{C}_\rho$ to keep track of the G -action.) There is an associated line bundle $L_\rho = [(V \times \mathbb{C}_\rho)/G]$ on $[V/G]$. Abusing notation, we also write L_ρ for restrictions of L_ρ to substacks $[V //_\theta G]$.

Proposition 2.3. *Let S be a scheme, and let $f : S \rightarrow [V/G]$ be a map, with corresponding principal G -bundle \mathcal{P} and map $\phi : \mathcal{P} \rightarrow V$. Then for any character ρ of G ,*

$$f^* L_\rho \cong \mathcal{P} \times_G \mathbb{C}_\rho \cong \sigma_\phi^*(\mathcal{P} \times_G (V \times \mathbb{C}_\rho)).$$

2.2. 3-stable curves and their line bundles.

Definition 2.4. An m -marked prestable orbifold curve (C, b_1, \dots, b_m) is a balanced twisted nodal m -pointed curve in the sense of [2]. That is, étale locally at each point P it is either:

- (1) isomorphic to $[\mathbb{C}/\mu_{d_P}]$ for some d_P , where μ_{d_P} acts by multiplication, and P is identified with 0, or
- (2) isomorphic to $[\mathbb{V}(xy)/\mu_{d_P}]$, where $\mathbb{V}(xy) \subseteq \mathbb{C}^2$ is the union of the coordinate axes, μ_{d_P} acts by multiplication by opposite roots of unity on x and y , and P is identified with $(0, 0)$,

together with m distinct marked points b_1, \dots, b_m of type (1), including all of those with $d_P > 1$. We refer to d_P as the *order* of P and μ_{d_P} as the *isotropy group* of P . We often write C instead of (C, b_1, \dots, b_m) .

Remark 2.5. For points P of type (1), there is a canonical identification of the isotropy group of P with μ_{d_P} ; the canonical generator is that which acts by multiplication by $e^{2\pi i/d_P}$ on $T_P C$. However, this is not true for points of type (2), since each element acts by opposite roots of unity on the two branches. Instead, there is a canonical identification *after* choosing a branch of the node.

Note that $d_P = 1$ for all but finitely many points P of C . An m -marked prestable orbifold curve admits a *coarse moduli space* map to an ordinary m -marked prestable curve \overline{C} . Olsson ([27]) proved that families of m -marked orbifold curves whose coarse moduli spaces have arithmetic genus g form an algebraic stack $\mathfrak{M}_g^{\text{tw}}$. We will only be interested in the case $g = 0$. For the purposes of this paper we restrict to an open substack of $\mathfrak{M}_0^{\text{tw}}$, as follows.

Definition 2.6 ([29]). A genus zero m -marked orbifold curve is *stable* if each irreducible component has at least three marked points or nodes. An m -marked *3-stable curve* is a stable genus zero m -marked orbifold curve such that all marked points and nodes are orbifold points of order 3.

Next we review some facts about line bundles on orbifold curves.

Definition 2.7. Let C be an m -marked prestable orbifold curve. A *line bundle* on C is a stack \mathcal{L} with a map to C , such that \mathcal{L} is étale locally isomorphic on C it is isomorphic to one of the following, corresponding to the cases in Definition 2.4:

- (1) $[\mathbb{C} \times \mathbb{C}/\mu_{d_P}]$, where μ_{d_P} acts by multiplication on the first copy of \mathbb{C} and linearly on the second copy, or

- (2) $[\mathbb{V}(xy) \times \mathbb{C}/\mu_{d_P}]$, where μ_{d_P} acts on $\mathbb{V}(xy)$ as in item (2) of Definition 2.4, and linearly on \mathbb{C} .

Definition 2.8. In case (1), $e^{2\pi i/d_P} \in \mu_{d_P}$ acts on the second copy of \mathbb{C} by multiplication by $e^{2\pi i k/d_P}$ for some $0 \leq k < d_P$. We call the rational number $\text{mult}_P(\mathcal{L}) := k/d_P$ the *multiplicity* or the *monodromy* of \mathcal{L} at P . If $\text{mult}_P(\mathcal{L}) = 0$, we say \mathcal{L} has *trivial monodromy* at P .

Remark 2.9. We also refer to the multiplicity of \mathcal{L} at a node of C . As in Remark 2.5, this is well-defined only after choosing a branch of the node. In this case we will refer to the multiplicity of \mathcal{L} “on one side of the node.”

One can similarly define vector bundles and their duals, sections, tensor products, and direct sums on orbifold curves. These behave largely the same as on nonstacky curves, with a few differences. For example, local sections of \mathcal{L} at an orbifold point P of C are μ_{d_P} -invariant sections as in (1) above, so in particular, if \mathcal{L} has nontrivial monodromy at P then every *local* section of \mathcal{L} vanishes at P . More specifically, if we define the order of vanishing of a section via pulling back along an étale cover by a scheme, then the order of vanishing of a section at an orbifold point P is an element of $\text{mult}_P(\mathcal{L}) + \mathbb{Z}$. Also, the monodromy of a tensor product of line bundles is given by

$$\text{mult}_P(\mathcal{L} \otimes \mathcal{L}') = \text{mult}_P(\mathcal{L}) + \text{mult}_P(\mathcal{L}') \pmod{1}.$$

Isomorphism classes of line bundles on orbifold curves may be easily understood via the *divisor-line bundle correspondence for smooth orbifold curves*. We state it only for genus zero curves, as these are all we consider.

Definition 2.10. A *Weil divisor* on a smooth orbifold curve is a (finite) formal sum $D = \sum_{P \in C} a_P P$ of points. The *degree* $\deg(D)$ of D is $\sum_{P \in C} \frac{a_P}{d_P}$, where d_P is the order of P . The degree is clearly additive under addition of Weil divisors.

The notion of rational equivalence of Weil divisors on orbifold curves is identical to that for schemes. For genus zero orbifold curves, it reduces to the following.

Definition 2.11. Two Weil divisors $D = \sum_{P \in C} a_P P$ and $D' = \sum_{P \in C} a'_P P$ on a genus zero smooth orbifold curve are *rationally equivalent* if for each $P \in C$ we have $a_P \equiv a'_P \pmod{d_P}$ for all P , and $\deg(D) = \deg(D')$. We write $[D]$ of the rational equivalence class of D .

In particular, the divisors $d_P P$ are rationally equivalent for all points P . We have the following correspondence:

Proposition 2.12 (See [34]). *The additive group of Weil divisors on a smooth orbifold curve C up to rational equivalence is naturally isomorphic to the group of line bundles on C up to isomorphism, with the operation of tensor product.*

As for schemes, the zeroes and poles of a rational section of a line bundle \mathcal{L} define a divisor, and this determines one direction of the correspondence. From this we see that the multiplicity $\text{mult}_P(\mathcal{L})$ is identified with the rational number $\frac{a_P}{d_P} \pmod{1}$. The correspondence also allows us to define the degree $\deg(\mathcal{L})$ of a line bundle, additive under tensor product.

We will often refer to the *log-canonical bundle* $\omega_{C, \log}$ on a 3-stable curve C . This is a line bundle whose sections are (correctly defined) holomorphic 1-forms on C , twisted by the divisor $\sum_i b_i$ of marked points. As with ordinary nodal curves, these holomorphic 1-forms may have simple poles at nodes. It will be important that $\omega_{C, \log}$ has trivial monodromy at each orbifold point, and has degree $2g - 2 + m = -2 + m$.

The curve $\mathbb{P}_{3,1} := [(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*]$, where \mathbb{C}^* acts with weights 3 and 1 on the coordinates respectively, will be particularly useful to us. This curve is smooth, and has a single orbifold marked point of order 3 at $[1 : 0]$, which we refer to as ∞ . (In particular, it is not 3-stable.) The group

of Weil divisor classes (and hence the group of isomorphism classes of line bundles) is generated by a single element $[\infty]$. Following convention we refer to the corresponding isomorphism class of line bundles as $\mathcal{O}_{\mathbb{P}_{3,1}}(1)$, and its tensor powers by $\mathcal{O}_{\mathbb{P}_{3,1}}(n)$. Note that $\deg(\mathcal{O}_{\mathbb{P}_{3,1}}(n)) = n/3$. The log canonical bundle $\omega_{\mathbb{P}_{3,1}, \log}$ (viewing $\mathbb{P}_{3,1}$ as a 1-marked orbifold curve) has degree $-2 + 1 = -1$, so it is isomorphic to $\mathcal{O}_{\mathbb{P}_{3,1}}(-3)$.

2.3. The inertia stack and Chen-Ruan cohomology. Let X be a smooth complex orbifold, i.e. a smooth connected Deligne-Mumford stack of finite type over \mathbb{C} .

Definition 2.13. Suppose $X = [M/G]$, where M is a smooth scheme and G is an abelian group. Then the *inertia stack* of X is the quotient $IX := [\tilde{M}/G]$, where \tilde{M} is the scheme parametrizing pairs (m, g) where $m \in M$ and $g \in G_m$, where $G_m \subseteq G$ is the stabilizer of m .

For fixed $g \in G$, let $\tilde{M}(g)$ be the open closed subscheme of \tilde{M} of elements of the form (m, g) . The *rigidified inertia stack* is the union

$$\bar{IX} := \bigcup_{g \in G} [\tilde{M}(g)/(G/\langle g \rangle)].$$

(Note: In the cases we consider, $\tilde{M}(g)$ is empty for all but finitely many g . More generally, the rigidified inertia stack is slightly more difficult to define.) The inertia stack and rigidified inertia stack are defined in much more generality (see [1]), but we will only need the cases above.

Remark 2.14. The rigidified inertia stack has the same coarse moduli space as the inertia stack. Indeed, the only difference between the two is that the inertia stack has “extra” stack structure. For example, $B\mu_3 := [\text{Spec } \mathbb{C}/\mu_3]$ has inertia stack $IB\mu_3 \cong B\mu_3 \sqcup B\mu_3 \sqcup B\mu_3$, and rigidified inertia stack $\bar{IB}\mu_3 \cong B\mu_3 \sqcup \text{Spec } \mathbb{C} \sqcup \text{Spec } \mathbb{C}$.

Terminology 2.15. Connected components of IX and \bar{IX} are called *sectors*. Both IX and \bar{IX} contain X as a connected component, namely the quotient $[\tilde{M}(e)/G]$ for $e \in G$ the identity. This component is referred to as the *untwisted sector* of IX or \bar{IX} , and other components are called *twisted sectors*. A twisted sector X' has a corresponding element $g \in G$, so we refer to X' as a *g-twisted sector*.

Remark 2.16. Note that there is a forgetful map $IX \rightarrow X$ that realizes each component of IX as a closed substack of X . There are also *inversion* automorphisms v on IX and \bar{IX} , which send $(m, g) \mapsto (m, g^{-1})$.

Definition 2.17. The *Chen-Ruan cohomology* of X is defined, as a \mathbb{C} -vector space, to be

$$H_{CR}^*(X) := H^*(\bar{IX}, \mathbb{C}),$$

where the right side denotes the singular cohomology of the coarse moduli space.

The grading of $H_{CR}^*(X)$ is different from the usual one. To describe it we use the following:

Definition 2.18. If X' is a g -twisted sector, then for generic $(m, g) \in X'$ we diagonalize the automorphism of $T_m M$ induced by g . The eigenvalues are roots of unity $e^{2\pi i \alpha_j}$ with $0 \leq \alpha_j < 1$. The *age* of X' , denoted $\text{age}(X')$, is defined to be $\sum_j \alpha_j$.

We may now define the grading:

$$H_{CR}^k(X) := \bigoplus_{X'} H^{k-2\text{age}(X')}(X', \mathbb{C}).$$

Remark 2.19. There is a natural graded embedding $H^*(X, \mathbb{C}) \hookrightarrow H_{CR}^*(X)$, induced by the inclusion $X \hookrightarrow IX$ of the untwisted sector.

Notation 2.20. For each sector X' , there is a class $1_{X'} \in H_{CR}^*(X)$ that is the unit in $H^*(X', \mathbb{C})$. (Its degree may be nonzero under the grading above.) If $\tilde{M}(g)$ is connected we will write $1_{X'} = 1_g$. We will also write $1_{X'}$ or 1_g for the corresponding class on the *nonrigidified* inertia stack.

Here are two important properties of Chen-Ruan cohomology:

- (1) There is a natural notion of cup product on $H_{CR}^*(X)$, compatible with the grading. (Note that the cup product on $H^*(IX, \mathbb{C})$ is not compatible with the grading as defined.)
- (2) If X is proper, there is a (perfect) Poincaré pairing on $H_{CR}^*(X)$, defined by $\langle \alpha, \beta \rangle_X = \int_{IX} \alpha \cup v^* \beta$. (The $v^* \beta$ in the integrand makes the pairing compatible with the grading.)

2.4. Cohomology of \mathbb{P}^2/μ_3 . One of the basic objects of this paper is the stack quotient $[\mathbb{P}^2/\mu_3]$, where μ_3 acts by multiplication on the first coordinate. In this section we consider only singular cohomology of the coarse moduli space \mathbb{P}^2/μ_3 , not Chen-Ruan cohomology. Write $[x_0 : x_1 : x_2]$ for points of \mathbb{P}^2 , and $\eta : \mathbb{P}^2 \rightarrow [\mathbb{P}^2/\mu_3]$ for the quotient map. We use the following notation: The

Symbol	Locus in \mathbb{P}^2	Symbol	Locus in $[\mathbb{P}^2/\mu_3]$
\tilde{L}_0	Line $\mathbb{V}(x_0) \subseteq \mathbb{P}^2$	L_0	$\eta(\tilde{L}_0)$
\tilde{P}_0	Point $[1 : 0 : 0] \in \mathbb{P}^2$	P_0	$\eta(\tilde{P}_0)$
\tilde{L}'	Any line through \tilde{P}_0	L'	$\eta(\tilde{L}')$
\tilde{P}'	Any point on \tilde{L}_0	P'	$\eta(\tilde{P}')$

μ_3 -fixed locus in \mathbb{P}^2 is $\tilde{L}_0 \cup \tilde{P}_0$. The lines \tilde{L}' are μ_3 -invariant, and these are the only μ_3 -invariant lines (other than \tilde{L}_0).

$H^*(\mathbb{P}^2/\mu_3, \mathbb{Z})$ is generated by $\{1, [L_0], [L'], [P_0], [P']\}$. Note that $3[L_0] = 3[L']$ and $3[P_0] = 3[P']$. Therefore we define the generators $H := [L_0] = [L']$ and $P := [P_0] = [P']$ of the *complex* cohomology $H^*(\mathbb{P}^2/\mu_3, \mathbb{C})$.

Remark 2.21. We usually consider not $[\mathbb{P}^2/\mu_3]$ but the line bundle $[\mathcal{O}_{\mathbb{P}^2}(-3)/\mu_3]$. Here $\mathcal{O}_{\mathbb{P}^2}(-3)$ is the quotient $((\mathbb{C}^3 \setminus \{(0, 0, 0)\}) \times \mathbb{C})/\mathbb{C}^*$, where \mathbb{C}^* acts with weights $(1, 1, 1, -3)$. The group μ_3 acts on the (quasi-)homogeneous coordinates of $\mathcal{O}_{\mathbb{P}^2}(-3)$ by $\zeta \cdot [x_0 : x_1 : x_2 : p_x] = [\zeta x_0 : x_1 : x_2 : p_x]$. The pullback map $H^*(\mathbb{P}^2/\mu_3, \mathbb{C}) \rightarrow H^*(\mathcal{O}_{\mathbb{P}^2}(-3)/\mu_3, \mathbb{C})$ is an isomorphism, and we will also use the symbols H and P to denote the corresponding classes in the latter.

3. THE TARGETS $Z(\theta)$

3.1. Notation. We begin by fixing notation that we will use throughout the paper. It is essentially in agreement with the notation of [21]. We let $V = \mathbb{C}^{13}$ with coordinates

$$(x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, a, p_x, p_y, p_z).$$

Let $G = (\mathbb{C}^*)^4$, with action on V by

$$\begin{aligned} (t_x, t_y, t_z, t_a) \cdot (x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, a, p_x, p_y, p_z) \\ = (t_x t_a^{-1} x_0, t_x x_1, t_x x_2, t_y t_a^{-1} y_0, t_y y_1, t_y y_2, t_z t_a^{-1} z_0, t_z z_1, t_z z_2, t_a^3 a, t_x^{-3} p_x, t_x^{-3} p_y, t_x^{-3} p_z). \end{aligned}$$

Define another group $\mathbb{C}_R^* = \mathbb{C}^*$ (denoted thus to avoid confusion) acting on V by

$$\begin{aligned} t_R \cdot (x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, a, p_x, p_y, p_z) \\ = (x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, a, t_R p_x, t_R p_y, t_R p_z). \end{aligned}$$

Remark 3.1. The groups G and \mathbb{C}_R^* are “independent” in that $\langle G, \mathbb{C}_R^* \rangle \subseteq \text{GL}(V)$ is isomorphic to $G \times \mathbb{C}_R^*$.

Let $W : V \rightarrow \mathbb{C}$ be the G -invariant function

$$W := p_x(ax_0^3 + x_1^3 + x_2^3) + p_y(ay_0^3 + y_1^3 + y_2^3) + p_z(az_0^3 + z_1^3 + z_2^3).$$

It is \mathbb{C}_R^* -homogeneous of degree 1.

Terminology 3.2. In the literature, W is referred to as a *superpotential* on V , and \mathbb{C}_R^* is known as an *R-charge*.

As $G \times \mathbb{C}_R^*$ acts diagonally, V is a direct sum of 1-dimensional $G \times \mathbb{C}_R^*$ -representations, corresponding to the list of characters

$$\mathbf{R} = \{\widehat{t}_x - \widehat{t}_a, \widehat{t}_x, \widehat{t}_x, \widehat{t}_y - \widehat{t}_a, \widehat{t}_y, \widehat{t}_y, \widehat{t}_z - \widehat{t}_a, \widehat{t}_z, \widehat{t}_z, 3\widehat{t}_a, -3\widehat{t}_x + \widehat{t}_R, -3\widehat{t}_y + \widehat{t}_R, -3\widehat{t}_z + \widehat{t}_R\},$$

where \widehat{t}_x is the character $(t_x, t_y, t_z, t_a, t_R) \mapsto t_x$, and similarly for the others.

The critical locus $\text{Crit}(W)$ of W is G -invariant. Let

$$\begin{aligned} X &:= [V/G] \\ Z &:= [\text{Crit}(W)/G], \end{aligned}$$

with $\iota : Z \hookrightarrow X$ the natural embedding. X and Z are nonseparated Artin stacks, but we consider certain open GIT quotient substacks, as follows.

3.2. The characters θ . We study GIT quotients $[V //_{\theta} G] = [V^{ss}(\theta)/G]$ where $\theta : G \rightarrow \mathbb{C}^*$ varies. The Euclidean space parametrizing $\theta : (t_x, t_y, t_z, t_a) \mapsto t_x^{e_x} t_y^{e_y} t_z^{e_z} t_a^{e_a}$ is isomorphic to $\mathbb{Z}^4 \otimes \mathbb{R} = \mathbb{R}^4$. The 16 GIT chambers are those on which the signs of e_x, e_y, e_z , and e_a are constant. We define characters $\Theta = \{\theta^{xyza}, \theta^{xya}, \theta^{xa}, \theta_{xyz}^a\}$ representing four of the chambers:

$$\begin{aligned} \theta^{xyza}(t_x, t_y, t_z, t_a) &= t_x^3 t_y^3 t_z^3 t_a^3 \\ \theta^{xya}(t_x, t_y, t_z, t_a) &= t_x^3 t_y^3 t_z^{-3} t_a^3 \\ \theta^{xa}(t_x, t_y, t_z, t_a) &= t_x^3 t_y^{-3} t_z^{-3} t_a^3 \\ \theta_{xyz}^a(t_x, t_y, t_z, t_a) &= t_x^{-3} t_y^{-3} t_z^{-3} t_a^3. \end{aligned}$$

(The multiples of 3 will simplify notation later.) We then define for $\theta \in \Theta$:

$$\begin{aligned} X(\theta) &:= [V //_{\theta} G] \subseteq X \\ Z(\theta) &= [\text{Crit}(W) //_{\theta} G] := [(\text{Crit}(W) \cap V^{ss}(\theta))/G] \subseteq Z. \end{aligned}$$

Again, we use ι to denote the embedding $Z(\theta) \hookrightarrow X(\theta)$.

Terminology 3.3. If x (resp. y, z) is in the subscript of θ , we will say “ x (resp. y, z) is a subscript variable.” Similarly we refer to “superscript variables,” and to modifying a character by “moving x from the superscript to the subscript.” (In every case, a is a superscript variable.)

Remark 3.4. As mentioned in the introduction, the characters θ^{xyza} and θ_{xyz}^a are of primary interest, as from them we will construct moduli spaces previously studied in Gromov-Witten theory and Fan-Jarvis-Ruan-Witten theory, respectively. The characters θ_z^{xya} and θ_{yz}^{xa} will provide a means of interpolating between these moduli spaces. By symmetry of x, y , and z , everything that follows regarding the characters in Θ works equally well for the characters $\theta_y^{xza}, \theta_x^{yza}, \theta_{xz}^{ya}$, and θ_{xy}^{za} .

Characters on walls of the chamber decomposition (such as $\theta_y^{xa}(t_x, t_y, t_z, t_a) := t_x^3 t_y^{-3} t_a^3$) are not considered. The corresponding GIT quotients are not well-behaved, and the moduli spaces of Section 4 are not defined in this situation.

The other missing characters are those where a is a subscript variable, e.g. $\theta_{za}^{xy}(t_x, t_y, t_z, t_a) := t_x^3 t_y^3 t_z^{-3} t_a^{-3}$. These are not needed to carry out the interpolation mentioned. However, this case may be of independent interest and we hope to return to it in the future.

Terminology 3.5. The GIT chambers are called *phases* in the physics literature, and various manifestations of GIT wall-crossing (such as the LG/CY correspondence) are known as *phase transitions*.

The following will help us state the definitions of moduli spaces in Section 4. Define characters

$$\begin{aligned}\vartheta^{xyza}(t_x, t_y, t_z, t_a) &= t_x^3 t_y^3 t_z^3 t_a^3 \\ \vartheta_z^{xya}(t_x, t_y, t_z, t_a) &= t_x^3 t_y^3 t_z^{-3} t_a^3 t_R \\ \vartheta_{yz}^{xa}(t_x, t_y, t_z, t_a) &= t_x^3 t_y^{-3} t_z^{-3} t_a^3 t_R^2 \\ \vartheta_{xyz}^a(t_x, t_y, t_z, t_a) &= t_x^{-3} t_y^{-3} t_z^{-3} t_a^3 t_R^3.\end{aligned}$$

of $G \times \mathbb{C}_R^*$. These lift the characters in Θ to $G \times \mathbb{C}_R^* \supseteq G$.

The GIT quotients $X(\theta)$ and $Z(\theta)$. A routine calculation of the equations defining $V^{\text{uns}}(\theta)$ and $\text{Crit}(W)$ yields the following characterization:

- (1) If x is a superscript variable of θ , then $\mathbb{V}(x_0, x_1, x_2) \subseteq V^{\text{uns}}(\theta)$. If x is a subscript variable of θ , then $\mathbb{V}(p_x) \subseteq V^{\text{uns}}(\theta)$. For every θ we have $\mathbb{V}(a) \subseteq V^{\text{uns}}(\theta)$. These three conditions entirely cut out $V^{\text{uns}}(\theta)$ in V .
- (2) $X(\theta^{xyza})$ is isomorphic to the total space of the rank 3 vector bundle $[\mathcal{O}_{\mathbb{P}^2}(-3)^3/\mu_3]$ over $[(\mathbb{P}^2)^3/\mu_3]$. Here μ_3 acts on each copy of \mathbb{P}^2 and $\mathcal{O}_{\mathbb{P}^2}(-3)$ as in Section 2.4. $Z(\theta^{xyza})$ is isomorphic to the complete intersection $[E^3/\mu_3]$ inside the zero section of this vector bundle, where E is the μ_3 -invariant elliptic curve $\mathbb{V}(x_0^3 + x_1^3 + x_2^3) \subseteq \mathbb{P}^2$.
- (3) $X(\theta_z^{xya}) \cong [(\mathcal{O}_{\mathbb{P}^2}(-3)^2 \times [\mathbb{C}^3/\mu_3])/\mu_3]$, where μ_3 acts on \mathbb{C}^3 by scaling. $Z(\theta_z^{xya}) \cong [(E^2 \times B\mu_3)/\mu_3]$, where $B\mu_3 \subseteq [\mathbb{C}^3/\mu_3]$ is the origin.
- (4) $X(\theta_{yz}^{xa}) \cong [(\mathcal{O}_{\mathbb{P}^2}(-3) \times [\mathbb{C}^3/\mu_3]^2)/\mu_3]$, and $Z(\theta_{yz}^{xa}) \cong [(E \times (B\mu_3)^2)/\mu_3]$.
- (5) $X(\theta_{xyz}^a) \cong [([\mathbb{C}^3/\mu_3]^3/\mu_3) \cong [\mathbb{C}^9/(\mu_3)^4]$, and $Z(\theta_{xyz}^a) \cong B((\mu_3)^4)$ is the origin.

Remark 3.6. Using e.g. the j -invariant, we may check that E is isomorphic to a quotient of \mathbb{C} by the lattice generated by $\{1, e^{2\pi i/6}\}$, and the μ_3 -action lifts to the multiplication action of μ_3 on \mathbb{C} . In this picture, we may identify $H^{1,0}(E)$ with $\mathbb{C}d\tau$, where τ is the coordinate on \mathbb{C} . The μ_3 -action on $H^{1,0}(E)$ is by multiplication.

Since we have $H^{3,0}(E^3) \cong H^{1,0}(E)^{\otimes 3}$, the diagonal μ_3 -action on $H^{3,0}(E^3)$ is trivial. In other words, the nonvanishing holomorphic 3-form on E^3 (unique up to scaling) is invariant under the μ_3 -action, so it descends to $Z(\theta^{xyza})$. That is, Z^{xyza} is Calabi-Yau.

Remark 3.7. In every case, \mathbb{C}_R^* acts trivially on $Z(\theta)$. For example, \mathbb{C}_R^* acts on $X(\theta^{xyza})$ by scaling on the fibers of the vector bundle $[(\mathcal{O}_{\mathbb{P}^2}(-3))^3/\mu_3]$, so acts trivially on $Z(\theta^{xyza})$ since $Z(\theta^{xyza})$ lies inside the zero section. Similarly, \mathbb{C}_R^* acts on $X(\theta_{xyz}^a)$ by scaling the coordinates of $[\mathbb{C}^9/\mu_3]$, so acts trivially on the origin.

Remark 3.8. We may check that for $\theta \in \Theta$, $V^{ss}(\theta)$ is equal to $V^{ss}(\vartheta)$, for ϑ the lift of θ defined above.

3.3. Toric divisors. We will often refer to the *toric divisors* $D_\rho \in H^2(X(\theta), \mathbb{C})$ and their pullbacks $\iota^* D_\rho \in H^2(Z(\theta), \mathbb{C})$. A character $\rho : G \rightarrow \mathbb{C}^*$ defines a line bundle L_ρ as in Section 2.1. For $\rho \in \mathbf{R}$, the corresponding coordinate s_ρ is a section of L_ρ . As usual, abusing notation we also write L_ρ and s_ρ for the restriction to each quotient $X(\theta) \subseteq X$. We define $D_\rho := c_1(L_\rho)$.

For $\theta = \theta_z^{xya}$, we compute D_ρ and $\iota^* D_\rho$ explicitly. Observe that $X(\theta)$ admits projection maps $\text{pr}_x, \text{pr}_y : X(\theta) \rightarrow [\mathcal{O}_{\mathbb{P}^2}(-3)/\mu_3]$ and $\text{pr}_z : X(\theta) \rightarrow [\mathbb{C}^3/(\mu_3)^2]$. We consider the vanishing loci in

$X(\theta)$ of the sections s_ρ . The sections $s_{\rho_{x_0}}, s_{\rho_{x_1}}, s_{\rho_{x_2}}$ are pulled back along pr_x . They cut out the fibers in $[\mathcal{O}_{\mathbb{P}^2}(-3)/\mu_3]$ over the coordinate lines in $[\mathbb{P}^2/\mu_3]$, and similarly for $s_{\rho_{y_0}}, s_{\rho_{y_1}}, s_{\rho_{y_2}}$. Using Section 2.4, these substacks give classes $[L_0], [L'], [L']$, respectively, and all of these are equal to H . Thus we have

$$\begin{aligned} D_{\rho_{x_0}} &= D_{\rho_{x_1}} = D_{\rho_{x_2}} = \text{pr}_x^*(H) =: H_x \\ D_{\rho_{y_0}} &= D_{\rho_{y_1}} = D_{\rho_{y_2}} = \text{pr}_y^*(H) =: H_y. \end{aligned}$$

The sections $s_{\rho_{z_0}}, s_{\rho_{z_1}}, s_{\rho_{z_2}}$ are pulled back along pr_z . They cut out the coordinate planes in $[\mathbb{C}^3/\mu_3]$. These are trivial in $H^2(X(\theta), \mathbb{C})$, i.e.

$$D_{\rho_{z_0}} = D_{\rho_{z_1}} = D_{\rho_{z_2}} = 0.$$

The section s_{ρ_a} is nonvanishing, so $D_{\rho_a} = 0$. By the same argument, $D_{\rho_{p_x}} = 0$. Finally, $s_{\rho_{p_x}}$ is again pulled back along pr_x , and vanishes along the zero section of $[\mathcal{O}_{\mathbb{P}^2}(-3)/\mu_3]$. Since $x_1^3 p_x$ is a well-defined function on $[\mathcal{O}_{\mathbb{P}^2}(-3)/\mu_3]$ (with coordinates as above), which vanishes to order 3 along $L' = \{x_1 = 0\}$ and to order 1 along the zero section. Hence the class of the zero section is $-3[L'] = -3H$, so

$$\begin{aligned} D_{\rho_{p_x}} &= -3H_x \\ D_{\rho_{p_y}} &= -3H_y. \end{aligned}$$

4. MODULI SPACES OF SECTIONS OF LINE BUNDLES

4.1. LG-quasimaps. For each $\theta \in \Theta$, we define moduli spaces of *LG-quasimaps*. These were introduced in [21], though in our examples the definitions simplify significantly.

Definition 4.1. A *prestable genus zero m -marked LG-quasimap* to $X(\theta)$ is a tuple (C, u, κ) , where

- (i) (C, b_1, \dots, b_m) is an m -marked 3-stable curve,
- (ii) $u : C \rightarrow [V/(G \times \mathbb{C}_R^*)] = [X/\mathbb{C}_R^*]$ is a morphism of stacks, and
- (iii) $\kappa : u^* L_{\widehat{t_R}} \rightarrow \omega_{C, \log}$ is an isomorphism of line bundles on C ,

such that all marked points, nodes, and generic points of components map to $[X(\theta)/\mathbb{C}_R^*] = [V //_{\vartheta} (G \times \mathbb{C}_R^*)]$. A *prestable genus zero m -marked LG-quasimap* to $Z(\theta)$ is a prestable genus zero m -marked LG-quasimap to $X(\theta)$ that factors through $[Z/\mathbb{C}_R^*] \hookrightarrow [X/\mathbb{C}_R^*]$.

Definition 4.2. Let (C, u, κ) be a prestable genus zero m -marked LG-quasimap to $X(\theta)$. A point P of C for which $u(P) \in [V^{un}(\theta)/(G \times \mathbb{C}_R^*)] \subseteq [V/(G \times \mathbb{C}_R^*)]$ is called a *basepoint* of u .

We will now reinterpret these definitions more algebraically. From Section 2.1, Definition 4.1(ii) is the same as a principal $(\mathbb{C}^*)^5$ -bundle \mathcal{P} on C — we will denote the five corresponding line bundles by $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z, \mathcal{L}_a, \mathcal{L}_R$, and their degrees by $\beta_x, \beta_y, \beta_z, \beta_a, \beta_R$ — and a section σ of

$$\begin{aligned} \mathcal{E} := \mathcal{P} \times_{(\mathbb{C}^*)^5} V &= (\mathcal{L}_x \otimes \mathcal{L}_a^*) \oplus \mathcal{L}_x \oplus \mathcal{L}_x \oplus (\mathcal{L}_y \otimes \mathcal{L}_a^*) \oplus \mathcal{L}_y \oplus \mathcal{L}_y \oplus (\mathcal{L}_z \otimes \mathcal{L}_a^*) \oplus \mathcal{L}_z \oplus \mathcal{L}_z \oplus \\ &\oplus \mathcal{L}_a^{\otimes 3} \oplus (\mathcal{L}_x^{\otimes -3} \otimes \mathcal{L}_R) \oplus (\mathcal{L}_y^{\otimes -3} \otimes \mathcal{L}_R) \oplus (\mathcal{L}_z^{\otimes -3} \otimes \mathcal{L}_R). \end{aligned}$$

Using (iii) we may forget about \mathcal{L}_R altogether and replace the data (ii) and (iii) with the data of the line bundles $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z, \mathcal{L}_a$ and a section σ of

$$\begin{aligned} \mathcal{E} &= (\mathcal{L}_x \otimes \mathcal{L}_a^*) \oplus \mathcal{L}_x \oplus \mathcal{L}_x \oplus (\mathcal{L}_y \otimes \mathcal{L}_a^*) \oplus \mathcal{L}_y \oplus \mathcal{L}_y \oplus (\mathcal{L}_z \otimes \mathcal{L}_a^*) \oplus \mathcal{L}_z \oplus \mathcal{L}_z \oplus \\ &\oplus \mathcal{L}_a^{\otimes 3} \oplus (\mathcal{L}_x^{\otimes -3} \otimes \omega_{C, \log}) \oplus (\mathcal{L}_y^{\otimes -3} \otimes \omega_{C, \log}) \oplus (\mathcal{L}_z^{\otimes -3} \otimes \omega_{C, \log}). \end{aligned}$$

We will write $\mathcal{L}_\rho := u^* L_\rho$ for the summands of \mathcal{E} , i.e.

$$\mathcal{E} = \bigoplus_{\rho \in \mathbf{R}} \mathcal{L}_\rho,$$

and

$$\sigma = (\sigma_{x_0}, \sigma_{x_1}, \sigma_{x_2}, \sigma_{y_0}, \sigma_{y_1}, \sigma_{y_2}, \sigma_{z_0}, \sigma_{z_1}, \sigma_{z_2}, \sigma_a, \sigma_{p_x}, \sigma_{p_y}, \sigma_{p_z}).$$

Thus an LG-quasimap (C, u, κ) to $X(\theta)$ is the same data as a tuple (C, \mathcal{L}, σ) , where \mathcal{L} is shorthand for $(\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z, \mathcal{L}_a)$. LG-quasimaps to $Z(\theta)$ are similarly reinterpreted, and we use the notations (C, u, κ) and (C, \mathcal{L}, σ) interchangeably.

Definition 4.3. Let (C, \mathcal{L}, σ) be an LG-quasimap to $X(\theta)$ or $Z(\theta)$. The *degree* of (C, \mathcal{L}, σ) is the tuple of rational numbers $\beta := (\beta_x, \beta_y, \beta_z, \beta_a)$. (We do not need to include β_R as it is necessarily equal to $-2 + m$.)

Notation 4.4. For an arbitrary character ρ of $G \times \mathbb{C}_R^*$, we define $\mathcal{L}_\rho := u^* L_\rho = \mathcal{P} \times_{G \times \mathbb{C}_R^*} \mathbb{C}_\rho$ and denote by β_ρ the degree of \mathcal{L}_ρ .

Definition 4.5. For $\epsilon \in \mathbb{Q}_{>0}$, we say an m -marked, genus-zero LG-quasimap to $Z(\theta)$ or $X(\theta)$ is ϵ -stable if

- (1) The *length* $\ell^\sigma(P)$ of σ at each point P is at most 1, and
- (2) $\omega_{C, \log} \otimes \mathcal{L}_\rho^\epsilon$ is ample.

For the general definition of length, see [21]. We describe it in the case $\theta = \theta_z^{xya}$, from which the other cases are clear. The length is a sum over components of the unstable locus $V^{\text{uns}}(\theta)$. If $(\sigma_{x_0}, \sigma_{x_1}, \sigma_{x_2}) = (0, 0, 0)$ at a point $P \in C$, then the minimum order of vanishing of these sections at P is the contribution to the $\ell^\sigma(P)$ from the component $\{(x_0, x_1, x_2) = (0, 0, 0)\}$ of $V^{\text{uns}}(\theta)$. We denote this contribution by $\ell_x^\sigma(P)$. The contribution $\ell_y^\sigma(P)$ is defined similarly. The contribution $\ell_z^\sigma(P)$ from the component $\{p_z = 0\}$ is even simpler — it is just the order of vanishing of σ_{p_z} at P . Similarly $\ell_a^\sigma(P)$ is the order of vanishing of σ_a at P . Finally, we define $\ell^\sigma(P) = \ell_x^\sigma(P) + \ell_y^\sigma(P) + \ell_z^\sigma(P) + \ell_a^\sigma(P)$.

Definition 4.6. Let (C, u, κ) be an LG-quasimap to $X(\theta)$ or $Z(\theta)$ of degree β , and let P be a basepoint of u . The *degree* $\beta(P) = (\beta_x(P), \beta_y(P), \beta_z(P), \beta_a(P))$ of the basepoint P is defined, for $\theta = \theta_z^{xya}$, by

$$(\beta_x(P), \beta_y(P), \beta_z(P), \beta_a(P)) := (\ell_x^\sigma(P), \ell_y^\sigma(P), \frac{-1}{3}\ell_z^\sigma(P), \frac{1}{3}\ell_a^\sigma(P)).$$

The reason for this definition is as follows. Restricting (C, u, κ) to $C \setminus P$ gives a section $\sigma|_{C \setminus P}$ of $\mathcal{E}|_{C \setminus P}$. There is a way (unique up to isomorphism) to extend $(\mathcal{E}|_{C \setminus P}, \sigma|_{C \setminus P})$ to (\mathcal{E}', σ') , where \mathcal{E}' is a vector bundle on C and $\sigma' \in H^0(C, \mathcal{E}')$, such that $\ell^{\sigma'}(P) = 0$. This bundle \mathcal{E}' is associated to a space of LG-quasimaps of degree $\beta - \beta(P)$. Therefore the degree of (C, u, κ) is equal to the degree “over the generic points of C ”, plus the sum of the degrees of all basepoints.

Remark 4.7. As in [13], we may also define ϵ -stability for ϵ equal to either of the symbols $0+$ and ∞ . A quasimap is $(0+)$ -stable if it is ϵ -stable for all ϵ sufficiently small, and ∞ -stable if it is ϵ -stable for all ϵ sufficiently large.

Theorem 4.8 ([21], Theorem 1.1.1). *For each $\theta \in \Theta$, $\epsilon \in [0+, \infty]$, $m \in \mathbb{Z}_{\geq 0}$ and $\beta \in (\frac{1}{3}\mathbb{Z})^4$, there is a finite type, separated Deligne-Mumford stack $\text{LGQ}_{0,m}^\epsilon(X(\theta), \beta)$ of families of ϵ -stable genus zero m -marked LG-quasimaps to $X(\theta)$ of degree β .*

There is also a finite type, separated, proper Deligne-Mumford stack $\text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)$ of families of ϵ -stable genus zero m -marked LG-quasimaps to $Z(\theta)$ of degree β .

Definition 4.9 (Graph spaces). We will often use the slightly modified moduli spaces $\text{LGQG}_{0,m}^\epsilon(X(\theta), \beta)$ and $\text{LGQG}_{0,m}^\epsilon(Z(\theta), \beta)$, called *graph spaces* or spaces of *LG-graph quasimaps*. They parametrize ϵ -stable m -marked genus zero LG-quasimaps *with a parametrized component*, i.e. a map $\tau : C \rightarrow \mathbb{P}^1$ of degree 1. The stability condition is then imposed only on (the closure of) $C \setminus \widehat{C}$, where \widehat{C} is the parametrized component.

4.2. Effective and extremal degrees.

Definition 4.10. We say (β, m) is θ -effective if $\text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)$ is nonempty.

Proposition 4.11. If (β, m) is effective, we have:

- $\beta_a \geq 0$,
- $\beta_x \geq 0$ if x is a superscript variable, and
- $\beta_x \leq \frac{m-2}{3}$ if x is a subscript variable.

(By symmetry these statements hold with x replaced by y or z .)

Proof. First, for all θ the condition in Definition 4.1, together with the fact that $\{a = 0\} \subseteq V^{\text{uns}}(\theta)$, implies that \mathcal{L}_a^3 has a global section that is nonvanishing at the generic point of each component of C . This implies $\beta_a \geq 0$.

If x is a superscript variable, the fact that $\{(x_0, x_1, x_2) = (0, 0, 0)\} \subseteq V^{\text{uns}}(\theta)$ shows that at least one of $\mathcal{L}_x \otimes \mathcal{L}_a^*$ and \mathcal{L}_x has nonnegative degree. Since $\beta_a \geq 0$ we have $\beta_x \geq 0$.

If x is a subscript variable, since $\{p_x = 0\} \subseteq V^{\text{uns}}$, we have that $\mathcal{L}_x^{\otimes -3} \otimes \omega_{C, \log}$ has nonnegative degree, i.e. $\beta_x \leq \frac{m-2}{3}$. \square

Corollary 4.12. Let β_ϑ be as in Notation 4.4. Whenever (β, m) is effective, we have $\beta_\vartheta \in \mathbb{Z}_{\geq 0}$.

Definition 4.13. Even if the conditions of Proposition 4.11 are satisfied, we may have $-2 + m + \epsilon\beta_\vartheta < 0$, in which case Condition 2 of Definition 4.5 is never satisfied. We call such tuples (β, m) *unstable*. Explicitly, (β, m) is unstable when

- (1) $m = 2$ and $\beta = \beta_0(\theta, 2)$, or
- (2) $m = 1$ and $\beta_\vartheta > 1/\epsilon$, or
- (3) $m = 0$ and $\beta_\vartheta > 2/\epsilon$.

Remark 4.14. It is the existence of unstable tuples that allows explicit calculation in Section 9.

Extremal Degrees. Let $\theta = \theta_z^{xya}$, as this is the example we work out in later sections. As we observed in Section 4.2, if β_x , β_y , or β_a is negative, the moduli space is empty. Also, if $\beta_z > \frac{m-2}{3}$, then the line bundle $\mathcal{L}_z^{-3} \otimes \omega_{C, \log}$ has negative degree, so the moduli space is empty for the same reason. Therefore we say that the pair $((0, 0, \frac{m-2}{3}, 0), m)$ is *extremal*, and write $\beta_0(\theta, m) := (0, 0, \frac{m-2}{3}, 0)$.

Observation 4.15. We will often use the fact that $\beta_0(\theta, m_1 + m_2) = \beta_0(\theta, m_1 + 1) + \beta_0(\theta, m_2 + 1)$.

Remark 4.16. It follows from Remark 4.12 that for (β, m) effective, we have $\beta_\vartheta = 0$ if and only if $(\beta, m) = (\beta_0(\theta, m), m)$ is extremal.

Definition 4.17. If C is irreducible and the “degree over the generic point” $\beta - \sum_P \beta(P)$ from Definition 4.6 is equal to $\beta_0(\theta, m)$, then we say C is contracted by u . Similarly we can say an irreducible component C' of C is *contracted*.

The extremal degree $\beta_0(\theta, m)$ will play essentially the same role for us as $\beta = 0$ does in Gromov-Witten theory, with matters complicated slightly by the fact that for $\theta \neq \theta^{xyza}$, $\beta_0(\theta, m)$ is a function of m .

4.3. Connections to quasimaps and spin structures. Given an ϵ -stable LG-quasimap to $Z(\theta)$, we may extract a quasimap to $Z(\theta)$ (in the sense of [7]) as follows.

Definition 4.18. By Remark 3.7, \mathbb{C}_R^* acts trivially on $\text{Crit}(W)$. This implies that

$$[Z/\mathbb{C}_R^*] \cong Z \times B\mathbb{C}_R^*$$

has a projection map pr to Z . For an LG-quasimap (C, u, κ) , the pair $(C, \text{pr} \circ u)$ is a prestable quasimap to $Z(\theta)$, called the *quasimap associated to* (C, u, κ) .

Remark 4.19. As \mathbb{C}_R^* acts nontrivially on $X(\theta)$, there is no way to extract a quasimap from a LG-quasimap (C, u, κ) to $X(\theta)$, unless u maps C into the locus $X_R(\theta) \subseteq [X(\theta)/\mathbb{C}_R^*]$ of points whose isotropy group contains \mathbb{C}_R^* . In this case we obtain a quasimap to $X_R^{\text{rig}}(\theta)$, the rigidification of $X_R(\theta)$ by \mathbb{C}_R^* (see [1]). From the \mathbb{C}_R^* -action on $X(\theta)$ we see that $X_R^{\text{rig}}(\theta)$ is isomorphic to

$$[(\mathbb{P}^2)^i \times B\mu_3^{3-i}]/\mu_3 \subseteq [(\mathcal{O}_{\mathbb{P}^2}(-3))^i \times [\mathbb{C}^3/\mu_3]^{3-i}]/\mu_3$$

for some i depending on θ , where \mathbb{P}^2 denotes the zero section of $\mathcal{O}_{\mathbb{P}^2}(-3)$ and $B\mu_3$ denotes the origin in $[\mathbb{C}^3/\mu_3]$. For instance, for $\theta = \theta_z^{xya}$ there is a quasimap to $X(\theta)$ associated to (C, \mathcal{L}, σ) exactly when $\sigma_{z_0} = \sigma_{z_1} = \sigma_{z_2} = \sigma_{p_x} = \sigma_{p_y} = 0$. The space $X_R^{\text{rig}}(\theta)$ will later allow us to reduce statements about LG-quasimaps to known facts about quasimaps.

Notice that the quasimap associated to (C, u, κ) captures information about the superscript variables. We may also extract complementary data related to the subscript variables, as follows:

Proposition 4.20. *If x is in the subscript of θ , then forgetting everything except for C , \mathcal{L}_x , and σ_{p_x} gives maps from $\text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)$ and $\text{LGQ}_{0,m}^\epsilon(X(\theta), \beta)$ to a space $\mathcal{R}_{m, \epsilon(-3\beta_x + m - 2)}^3$ of (prestable) spin structures, see [11].*

If there are several subscript variables, this gives maps to a product of spaces of spin structures, fibered over $\overline{\mathcal{M}}_{0,m}$.

This follows from the following lemma, adapted from [29].

Lemma 4.21. *If x is in the subscript of θ , then if \mathcal{L}_x and $\mathcal{L}_x \otimes \mathcal{L}_a^*$ have nontrivial monodromy at each marked point, they have no global sections.*

Proof. The proof of Lemma 1.5 of [29] immediately generalizes to any line bundle on a nodal genus zero twisted curve such that a tensor power is a twist down of $\omega_{C, \log}$ by a effective divisor. \square

Remark 4.22. The assumption of nontrivial monodromy is very important, and we will later define numerical invariants to vanish when this assumption is not satisfied. (See Sections 5.1 and 5.2.)

Remark 4.23. In the case $\theta = \theta_{xyz}^a$ one can straightforwardly mimic the entire argument of [29], which proves the analog of Theorem 8.1 for a different class of moduli spaces coming from hypersurfaces in weighted projective spaces. Indeed for θ_{xyz}^a , Sections 6 and 8 are extremely simple, since the torus action of Section 6.2 is trivial.

Consider the space $\text{LGQ}_{0,m}^\epsilon(Z(\theta^{xyza}), \beta)$. Let (C, \mathcal{L}, σ) be an ϵ -stable genus-zero m -marked LG-quasimap to $Z(\theta)$. As $\sigma_{p_x} = \sigma_{p_y} = \sigma_{p_z} = 0$ by the condition that σ land in $Z(\theta)$, we may rephrase the data of (C, \mathcal{L}, σ) once again, and we arrive at exactly the data of an ϵ -stable quasimap (in the sense of [13]) to $Z(\theta)$. That is, we have

$$\text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta) = Q_{0,m}^\epsilon(Z(\theta), \beta),$$

where the latter space is a moduli stack of quasimaps. (The β on the right must be reinterpreted slightly as a character of G .) This isomorphism is, of course, the motivation for Definition 4.1. Note, however, that $\text{LGQ}_{0,m}^\epsilon(X(\theta), \beta)$ is *not* isomorphic to $Q_{0,m}^\epsilon(X(\theta), \beta)$.

One may wonder why, in this setup, we express the orbifold $[(\mathcal{O}_{\mathbb{P}^2}(-3))^3/\mu_3]$ as the complicated toric variety $X(\theta^{xyza}) = [\mathbb{C}^{13} // (\mathbb{C}^*)^4]$, rather than the more natural-seeming (and isomorphic) toric variety

$$[\mathbb{C}^{12} // ((\mathbb{C}^*)^3 \times \mu_3)].$$

The reason is a slightly complicated one. In fact, we could have used either presentation. However, we will later calculate an important generating function $I^\theta(q, z)$, which is defined using the moduli space $\text{LGQG}_{0,1}^{0+}(Z(\theta), \beta)$. This space parametrizes LG-quasimaps (C, u, κ) to $Z(\theta)$, where $C \cong \mathbb{P}_{3,1}$.

One can easily see from the definitions in [21] that there is no change if $\mathbb{P}_{3,1}$ is replaced with \mathbb{P}^1 ; in other words, the stack structure plays no role! The reason is that part of the data of u is a principal μ_3 -bundle on C , and a principal μ_3 -bundle on $\mathbb{P}_{3,1}$ is trivial, since the orbifold fundamental group of $\mathbb{P}_{3,1}$ is trivial.

This issue disappears in our setup, essentially because the line bundle \mathcal{L}_a may still be nontrivial for $(C, \mathcal{L}, \sigma) \in \text{LGQG}_{0,1}^{0+}(Z(\theta), \beta)$. As a result, $I^\theta(q, z)$ contains much less information when using the “natural” presentation of $X(\theta^{xyz a})$ than it does when using our presentation. This method of finding more informative presentations of orbifolds is alluded to in [12], and is related to the notion of *S-extended I-functions* from [16].

5. EVALUATION MAPS, COMPACT TYPE STATE SPACE, AND INVARIANTS

Evaluation maps. The universal curve $\text{UQ}_{0,m}^\epsilon(Z(\theta), \beta)$ over $\text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)$ has a universal map u^{rig} to Z , by Definition 4.18. By the condition in Definition 4.1, all marked points and (relative) nodes of $\text{UQ}_{0,m}^\epsilon(Z(\theta), \beta)$ map to $Z(\theta) \subseteq Z$. Thus by Section 4.4 of [1], there are *evaluation maps* $\text{ev}_i : \text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta) \rightarrow \bar{I}Z(\theta)$, the rigidified inertia stack of $Z(\theta)$, recording the image of the i th marked point. In [21] there is a more general (and more subtle) notion of evaluation map defined on $\text{LGQ}_{0,m}^\epsilon(X(\theta), \beta)$, and taking values in $\bar{I}X(\theta)$. There are also evaluation maps on graph spaces:

$$\text{ev}_i : \text{LGQG}_{0,m}^\epsilon(X(\theta), \beta) \rightarrow \bar{I}X(\theta) \times \mathbb{P}^1.$$

5.1. Compact type state space. Part of the setup of the gauged linear sigma model in [21] is a special graded vector space $\mathcal{H}(\theta)$, with a pairing, called the *state space*. It is defined via *relative* Chen-Ruan cohomology groups. In our case, we may work with a particularly simple subspace $\mathcal{H}(\theta)$.

Definition 5.1. A sector of $\bar{I}X(\theta)$ is called *narrow*² if for each subscript variable, the corresponding sector of $[\mathbb{C}^3/\mu_3]$ is compact, i.e. is isomorphic to $B\mu_3$. We denote the (open and closed) substack of narrow sectors by $\bar{I}X(\theta)^{\text{nar}} \subseteq \bar{I}X(\theta)$. The cohomology classes of narrow sectors are a direct summand $H_{CR}^*(X(\theta))^{\text{nar}} \subseteq H_{CR}^*(X(\theta))$.

Definition 5.2. The *ambient narrow state space* $\mathcal{H}(\theta)$ is the image $\iota^*(H_{CR}^*(X(\theta))^{\text{nar}}) \subseteq H_{CR}^*(Z(\theta))$.

Remark 5.3. The Poincaré pairing on $H_{CR}^*(Z(\theta))$ descends to $\mathcal{H}(\theta)$.

$\mathcal{H}(\theta)$ inherits a grading from $H_{CR}^*(Z(\theta))$, but it is not the correct one for our purposes. For example, for $\theta = \theta_{xyz}^a$ this would result in a vector space concentrated in degree zero, due to the fact that $Z(\theta) \cong [\text{Spec } \mathbb{C}/(\mu_3)^4]$. In fact, this issue also arises when trying to define a graded pullback map on Chen-Ruan cohomology. Instead, we define the grading as follows.

First, the ages of elements of $H_{CR}^*(Z(\theta))$ are calculated from the normal bundle to the embedding of a g -twisted sector not into $Z(\theta)$, but into the ambient space $X(\theta)$. (These are equivalent in the case where $Z(\theta)$ intersects the g -fixed locus in $X(\theta)$ transversely, which is the case for $\theta = \theta^{xyz a}$. However, it is not true for $\theta = \theta_{xyz}^a$, where $Z(\theta)$ is contained in every g -fixed locus.)

Second, we add to each degree the somewhat mysterious shift $\dim(Z(\theta)) - 3$. We do this in order to obtain the following:

Theorem 5.4. *There is a graded isomorphism between $\mathcal{H}(\theta)$ and $\mathcal{H}(\theta')$ for any $\theta, \theta' \in \Theta$. In particular, the graded subspaces have the dimensions:*

$$(1) \quad \begin{array}{c|c|c|c} \mathcal{H}^0(\theta) & \mathcal{H}^2(\theta) & \mathcal{H}^4(\theta) & \mathcal{H}^6(\theta) \\ \hline 1 & 4 & 4 & 1 \end{array}$$

²Note that our definition of a narrow sector is different from that in [21]; however, it is the correct one for this setting.

Remark 5.5. Proving this is just a calculation; we call it a theorem because it is one of the parts of Ruan's original LG/CY conjecture ([30]). There is a method ([9]) for proving more general statements of this form, via careful use of an orbifold Thom isomorphism theorem, but it does not yet apply to this case.

Proof. We record the enlightening parts of the proof here.

Compact type state space of $Z(\theta^{xyza})$. Let $\theta = \theta^{xyza}$. The inclusion $\iota : Z(\theta) \hookrightarrow X(\theta)$ factors as $\iota' \circ \iota''$, where ι' is the inclusion $[(\mathbb{P}^2)^3/\mu_3] \hookrightarrow X(\theta)$. Since ι' is a homotopy equivalence, we have $\text{Im}(\iota^*) \cong \text{Im}((\iota'')^*)$.

The points of $(\mathbb{P}^2)^3$ with nontrivial stabilizer are those points (p_1, p_2, p_3) , where $p_1, p_2, p_3 \in \mathbb{P}^2$ are all fixed by multiplication of the first coordinate by ζ . In other words, using the notation of Section 2.4, $p_i \in \tilde{L}_0$ or $p_i = \tilde{P}_0$. From this we see that the orbifold locus in $[(\mathbb{P}^2)^3/\mu_3]$ is a union of eight components. The seven components isomorphic to $L_0 \times L_0 \times P_0$, $L_0 \times P_0 \times P_0$, and $P_0 \times P_0 \times P_0$ do not intersect the critical locus, since $\tilde{P}_0 \notin E$. The component $L_0 \times L_0 \times L_0$ intersects $[E^3/\mu_3]$ in $3^3 = 27$ points (each isomorphic to $B\mu_3$). That is,

Proposition 5.6. *The rigidified inertia stack $\bar{I}Z(\theta)$ of $Z(\theta)$ is isomorphic to the disjoint union of $Z(\theta)$ and $27 \cdot 2 = 54$ points.*

It is easy to check that a \mathbb{C} -basis of $\text{Im}((\iota'')^*)$ consists of the pullbacks of the classes

$$\{1, H_x, H_y, H_z, H_x H_y, H_x H_z, H_y H_z, H_x H_y H_z\}.$$

The images of the classes $1_\zeta, 1_{\zeta^2} \in H_{CR}^*(X(\theta))$ are the sums of the 27 ζ -twisted and ζ^2 -twisted classes, respectively, in $H_{CR}^*(Z(\theta))$. The corresponding ages are 1 and 2, respectively. We thus obtain the table (1) for θ^{xyza} for $\mathcal{H}(\theta^{xyza})$.

Notation 5.7. We refer to e.g. $(\iota'')^*(H_y H_z)$ by the more-cumbersome notation $(1 \otimes H_y \otimes H_z)_1$, and denote $(\iota'')^*(1_\zeta)$ and $(\iota'')^*(1_{\zeta^2})$ by $(1 \otimes 1 \otimes 1)_\zeta$ and $(1 \otimes 1 \otimes 1)_{\zeta^2}$, respectively, since it will simplify our notation in the remaining cases.

Compact type state space of $Z(\theta_{xyz}^a)$. Let $\theta = \theta_{xyz}^a$. Recall from Section 3.2 that $X(\theta) \cong [\mathbb{C}^9/(\mu_3)^4]$, with $Z(\theta) = [pt/(\mu_3)^4]$ the origin. As all fixed loci are connected, the components of $\bar{I}X(\theta)$ and $\bar{I}Z(\theta)$ are both in bijection with $(\mu_3)^4$. The pullback map ι^* is surjective, and the narrow sectors correspond to elements of $(\mu_3)^4$ that act trivially on \mathbb{C}^9 . We can easily write down these elements, and calculating their ages gives

Element	Age
$(\zeta, \zeta, \zeta, 1)$	3
$(\zeta^2, \zeta, \zeta, 1)$	4
$(\zeta, \zeta^2, \zeta, 1)$	4
$(\zeta, \zeta, \zeta^2, 1)$	4
$(\zeta^2, \zeta^2, \zeta, 1)$	5
$(\zeta^2, \zeta, \zeta^2, 1)$	5
$(\zeta, \zeta^2, \zeta^2, 1)$	5
$(\zeta^2, \zeta^2, \zeta^2, 1)$	6
$(\zeta^2, \zeta^2, \zeta^2, \zeta)$	5
$(\zeta, \zeta, \zeta, \zeta^2)$	4

We denote the class associated to $(\zeta, \zeta, \zeta, 1)$ by $(1_\zeta \otimes 1_\zeta \otimes 1_\zeta)_1$, and similarly for the first eight rows of this list. The last two we denote $(1_{\zeta^2} \otimes 1_{\zeta^2} \otimes 1_{\zeta^2})_\zeta$ and $(1_\zeta \otimes 1_\zeta \otimes 1_\zeta)_{\zeta^2}$, respectively. The shifted degree of a class γ is $2 \text{ age}(\gamma) - 2 \dim Z(\theta) - 6 = 2 \text{ age}(\gamma) - 6$. Thus again we obtain (1).

Compact type state space of $Z(\theta_z^{xya})$. Finally, we include the computation for $\mathcal{H}^*(\theta)$ where $\theta = \theta_z^{xya}$, because this case will be worked out in detail throughout the paper.

Recall that

$$\begin{aligned} X(\theta) &\cong [(\mathcal{O}_{\mathbb{P}^2}(-3))^2 \times [\mathbb{C}^3/\mu_3]]/\mu_3 \\ Z(\theta) &\cong [(E^2 \times B\mu_3)/\mu_3] = [E^2/(\mu_3)^2]. \end{aligned}$$

The elements $(1, 1), (1, \zeta), (1, \zeta^2), (\zeta, \zeta), (\zeta^2, \zeta^2) \in (\mu_3)^2$ do not give narrow sectors. We write the narrow sectors associated to the other elements of $(\mu_3)^2$:

Group element	Sectors
$(\zeta, 1)$	$[E^2/\mu_3]$
$(\zeta^2, 1)$	$[E^2/\mu_3]$
(ζ^2, ζ)	9 points
(ζ, ζ^2)	9 points

The image of the pullback is calculated in the same way as it was for θ^{xyza} . In particular, we have a basis for $\mathcal{H}(\theta)$:

$$\begin{aligned} &\{(1 \otimes 1 \otimes 1_\zeta)_1, (H_x \otimes 1 \otimes 1_\zeta)_1, (1 \otimes H_y \otimes 1_\zeta)_1, (H_x \otimes H_y \otimes 1_\zeta)_1, \\ &\quad (1 \otimes 1 \otimes 1_{\zeta^2})_1, (H_x \otimes 1 \otimes 1_{\zeta^2})_1, (1 \otimes H_y \otimes 1_{\zeta^2})_1, (H_x \otimes H_y \otimes 1_{\zeta^2})_1, \\ &\quad (1 \otimes 1 \otimes 1_{\zeta^2})_\zeta, (1 \otimes 1 \otimes 1_\zeta)_{\zeta^2}\}. \end{aligned}$$

Here $(1 \otimes 1 \otimes 1_{\zeta^2})_\zeta$ is the sum of the first set of 9 points above, and $(H_x \otimes 1 \otimes 1_\zeta)_1$ is the pullback of the class H_x on the $(\zeta, 1)$ -twisted sector of $X(\theta)$, isomorphic to $[\mathcal{O}_{\mathbb{P}^2}(-3)^2/\mu_3]$. Calculating the (properly shifted) degrees gives (1) again.

It is straightforward to carry out the calculation for θ_{yz}^{xa} , with the same result. This proves the theorem. \square

Remark 5.8. This graded isomorphism holds for the larger state spaces $\mathcal{H}(\theta)$ defined in [21] as well.

Explicit isomorphisms. In Section 10, use an explicit identification of $\mathcal{H}(\theta)$ with $\mathcal{H}(\theta')$, which we describe here. Let θ be such that x is a superscript variable, and let θ' be the character obtained by moving x to the subscript. Elements of $\mathcal{H}(\theta)$ are of the form $(1 \otimes \alpha_y \otimes \gamma_z)_g$ or $(H_x \otimes \alpha_y \otimes \gamma_z)_g$ with $g \in \mu_3$. We send:

$$\begin{aligned} (1 \otimes \alpha_y \otimes \gamma_z)_g &\mapsto (1_\zeta \otimes \alpha_y \otimes \gamma_z)_g \in \mathcal{H}(\theta') \\ (H_x \otimes \alpha_y \otimes \gamma_z)_g &\mapsto (1_{\zeta^2} \otimes \alpha_y \otimes \gamma_z)_g \in \mathcal{H}(\theta'). \end{aligned}$$

Repeating this process and its inverse gives explicit graded isomorphisms between $\mathcal{H}(\theta)$ and $\mathcal{H}(\theta')$ for any $\theta, \theta' \in \Theta$.

Remark 5.9. For each θ , there is a special generator with degree zero. This is the element where $g = 1 \in \mu_3$ and all entries of the tensor are 1 or 1_ζ , for x a superscript or subscript variable respectively. We will abbreviate it by 1_θ .

5.2. Virtual class and invariants. We define open and closed substacks:

$$\begin{aligned} \text{LGQ}_{0,m}^\epsilon(X(\theta), \beta)^{\text{nar}} &:= \bigcap_{i=1}^m \text{ev}_i^{-1}(\bar{I}X(\theta)^{\text{nar}}) \subseteq \text{LGQ}_{0,m}^\epsilon(X(\theta), \beta) \\ \text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)^{\text{nar}} &:= \bigcap_{i=1}^m \text{ev}_i^{-1}(\bar{I}Z(\theta)^{\text{nar}}) \subseteq \text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta). \end{aligned}$$

Theorem 5.10 ([21]). *The complex $R^\bullet \pi_* \mathcal{E}$ is a perfect obstruction theory on $\mathrm{LGQ}_{0,m}^\epsilon(X(\theta), \beta)^{\mathrm{nar}}$. It induces (via cosection localization, see [25, 21]) a virtual fundamental class*

$$[\mathrm{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)^{\mathrm{nar}}]^{\mathrm{vir}} \in H_*([\mathrm{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)^{\mathrm{nar}}, \mathbb{C})).$$

There is similarly a virtual fundamental class on each graph space $\mathrm{LGQG}_{0,m}^\epsilon(Z(\theta), \beta)^{\mathrm{nar}}$.

By a general fact about cosection localization,

$$\iota_*[\mathrm{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)^{\mathrm{nar}}]^{\mathrm{vir}} = [\mathrm{LGQ}_{0,m}^\epsilon(X(\theta), \beta)^{\mathrm{nar}}]^{\mathrm{vir}},$$

where the latter is the virtual fundamental induced by the perfect obstruction theory $R^\bullet \pi_* \mathcal{E}$.

Using this, we may define *LG-quasimap invariants*:

Definition 5.11. Let $\alpha_1, \dots, \alpha_m \in \mathcal{H}(\theta)$. Then we define

$$\begin{aligned} \langle \alpha_1 \psi^{a_1}, \dots, \alpha_m \psi^{a_m} \rangle_{0,m,\beta}^{\epsilon,\theta} &:= \int_{[\mathrm{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)^{\mathrm{nar}}]^{\mathrm{vir}}} \prod_{i=1}^m (\psi_i^{a_i} \mathrm{ev}_i^* \alpha_i) \\ \langle \alpha_1 \psi^{a_1}, \dots, \alpha_m \psi^{a_m} \rangle_{0,m,\beta}^{\epsilon,\theta, Gr} &:= \int_{[\mathrm{LGQG}_{0,m}^\epsilon(Z(\theta), \beta)^{\mathrm{nar}}]^{\mathrm{vir}}} \prod_{i=1}^m (\psi_i^{a_i} \mathrm{ev}_i^* \alpha_i). \end{aligned}$$

Remark 5.12. Theorem 5.10 also states that the *unshifted virtual dimension* of $\mathrm{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)^{\mathrm{nar}}$ is m . This implies that if $\alpha_1, \dots, \alpha_m$ have degrees k_1, \dots, k_m , then $\langle \alpha_1 \psi^{a_1}, \dots, \alpha_m \psi^{a_m} \rangle_{0,m,\beta}^{\epsilon,\theta}$ vanishes unless $\sum_i (k_i + a_i) = m$.³ Similarly $\mathrm{LGQG}_{0,m}^\epsilon(Z(\theta), \beta)^{\mathrm{nar}}$ has unshifted virtual dimension $m + 3$.

Remark 5.13. We may define these invariants for arbitrary $\alpha \in H_{CR}^*(Z(\theta))$, with the convention that they vanish if α_i is supported on $\bar{I}X(\theta) \setminus \bar{I}X(\theta)^{\mathrm{nar}}$ for some i . (We refer to such α_i as *broad*, as we used *narrow* to refer to elements of $\mathcal{H}(\theta)$.)

6. EQUIVARIANT LOCALIZATION ON $\mathrm{LGQG}_{0,m}^\epsilon(X(\theta), \beta)$ AND $\mathrm{LGQ}_{0,m}^\epsilon(X(\theta), \beta)$

In this section we define two natural group actions on the moduli spaces; a \mathbb{C}^* -action on $\mathrm{LGQG}_{0,m}^\epsilon(X(\theta), \beta)$ induced by the \mathbb{C}^* -action on P^1 , and a torus action on $\mathrm{LGQ}_{0,m}^\epsilon(X(\theta), \beta)$ induced by the torus action on the toric variety $X(\theta)$.

6.1. The \mathbb{C}^* -action on $\mathrm{LGQG}_{0,m}^\epsilon(X(\theta), \beta)$. The \mathbb{C}^* -action we define, as well as the graph spaces themselves, are essentially combinatorial tools for analyzing the generating functions defined in Section 7. Recall that an LG-graph quasimap to $X(\theta)$ is a tuple $(C, \mathcal{L}, \sigma, \tau)$, where $\tau : C \rightarrow \mathbb{P}^1$ is a degree-one (nonrepresentable) map. For $\lambda \in \mathbb{C}^*$, let $\lambda \cdot [s : t] = [\lambda s : t]$ denote the standard (left) action on \mathbb{P}^1 . Then

$$(C, \mathcal{L}, \sigma, \tau) \mapsto (C, \mathcal{L}, \sigma, \tau \circ \lambda)$$

is a (right) \mathbb{C}^* -action on $\mathrm{LGQG}_{0,m}^\epsilon(X(\theta), \beta)$.

\mathbb{C}^* -fixed locus and normal bundles. An LG-graph quasimap $(C, \mathcal{L}, \sigma, \tau)$ to $X(\theta)$ is \mathbb{C}^* -fixed if for each $\lambda \in \mathbb{C}^*$ there exists an automorphism ϕ of C commuting with σ such that $\phi \circ \tau = \tau \circ \lambda$. Alternatively, let $\widehat{C}^\circ := C \setminus \{\tau^{-1}(0), \tau^{-1}(\infty)\}$ and $\widehat{C} := \widehat{C}^\circ \subseteq C$. Then $(C, \mathcal{L}, \sigma, \tau)$ is \mathbb{C}^* -fixed if

- (1) \widehat{C}° contains no marked points, nodes, or basepoints of σ , and
- (2) \widehat{C} is contracted by u .

³The term *unshifted virtual dimension* is nonstandard, and we define it by this property. In the statement in [21], the condition instead would read $\sum_i (k_i + a_i - \mathrm{age}_i) = m - \sum \mathrm{age}_i$, where age_i is the age of the twisted sector in which ev_i lands.

Notation 6.1 (From [29]). We denote by \widehat{C} the closure of \widehat{C}° , and we write C_0 and C_∞ for $\tau^{-1}(0)$ and $\tau^{-1}(\infty)$, respectively. We write $\bullet := C_0 \cap \widehat{C}$ and $\check{\bullet} := C_\infty \cap \widehat{C}$. The point \bullet may be a smooth (possibly orbifold) point (in which case C_0 is a single point), or it may be a node (in which case C_0 is a nodal curve).

Proposition 6.2. *A \mathbb{C}^* -fixed m -marked LG-quasimap $(C, \mathcal{L}, \sigma, \tau)$ to $X(\theta)$ of degree β defines a partition $B_0 \sqcup B_\infty$ of $\{1, \dots, m\}$ and a partition of tuples $\beta^0 + \beta^\infty = \beta$, such that $(\beta^0, |B_0| + 1)$ and $(\beta^\infty, |B_\infty| + 1)$ are θ -effective or unstable.*

Proof. The fact that $\tau(b_i)$ is either 0 or ∞ for each i defines a partition $\{1, \dots, m\} = B_0 \sqcup B_\infty$.

If \bullet is a node, then $(C_0, \mathcal{L}|_{C_0}, \sigma|_{C_0})$ is an ϵ -stable LG-quasimap to $X(\theta)$. (Here C_0 has the extra marking \bullet .) We denote by β^0 its degree, and similarly β^∞ . If there is no node at \bullet (resp. $\check{\bullet}$), then we define β^0 to be the degree of the basepoint at \bullet (resp. $\check{\bullet}$, see Definition 4.6).

If there are nodes at \bullet and $\check{\bullet}$, we can check that $\widehat{\beta} = 0$, so $\beta_0 + \beta^\infty = \beta$. If one or both of \bullet and $\check{\bullet}$ is not a node, $\beta_0 + \beta^\infty$ by the definition of the degree of a basepoint. \square

Remark 6.3. The tuple (B_0, β^0) is unstable exactly when \bullet is a smooth point.

Proposition 6.2 allows us to define open and closed substacks $F_{B_0, \beta^0}^{B_\infty, \beta^\infty}$ of $\text{LGQG}_{0,m}^\epsilon(X(\theta), \beta)^{\mathbb{C}^*}$, consisting of those LG-quasimaps that induce the partition $B_0 \sqcup B_\infty$ of $\{1, \dots, m\}$ and the partition $\beta^0 + \beta^\infty$ of β . (We refer to these as “components” of $\text{LGQG}_{0,m}^\epsilon(X(\theta), \beta)^{\mathbb{C}^*}$, though they are almost never connected.) If $(\beta^0, |B_0| + 1)$ and $(\beta^\infty, |B_\infty| + 1)$ are effective rather than unstable, then

$$(2) \quad F_{B_0, \beta^0}^{B_\infty, \beta^\infty} \cong \text{LGQ}_{0, |B_0| + \bullet}^\epsilon(X(\theta), \beta^0) \times_{\bar{I}Z(\theta)} \text{LGQ}_{0, |B_\infty| + \check{\bullet}}^\epsilon(X(\theta), \beta^\infty),$$

fibered over the evaluation maps ev_\bullet and $\text{ev}_{\check{\bullet}}$.⁴

When \bullet and $\check{\bullet}$ are both nodes, we may calculate the \mathbb{C}^* -equivariant Euler class of the virtual normal bundle to $F_{B_0, \beta^0}^{B_\infty, \beta^\infty}$. One may check that the \mathbb{C}^* -moving infinitesimal deformations come from smoothing the nodes and deforming the map τ (equivalently, moving the points $\tau(\bullet)$ and $\tau(\check{\bullet})$). By a classical computation, smoothing the nodes contributes factors $\hbar - \psi_\bullet$ and $-\hbar - \psi_{\check{\bullet}}$, pulled back to the fiber product (2). (These are the weights of the deformation spaces $T_\bullet C_0 \otimes T_\bullet \widehat{C}$ and $T_{\check{\bullet}} C_\infty \otimes T_{\check{\bullet}} \widehat{C}$, respectively. Here we use the natural identification $H_{\mathbb{C}^*}^*(\text{Spec } \mathbb{C}, \mathbb{C}) \cong \mathbb{C}[[\hbar]]$.) Deforming τ gives factors \hbar and $-\hbar$, the weights of the tangent spaces $T_0 \mathbb{P}^1$ and $T_\infty \mathbb{P}^1$. Thus the \mathbb{C}^* -equivariant Euler class of the virtual normal bundle is $(-\hbar^2)(\hbar - \psi_\bullet)(-\hbar - \psi_{\check{\bullet}})$.

Definition 6.4. We define here a special component $F'_\beta := F_{\star, \beta_0(2)}^{m, \beta}$ of $\text{LGQG}_{0, m+\star}^\epsilon(X(\theta), \beta)^{\mathbb{C}^*}$, which we will use in Sections 7.2 and 9.

Definition 6.5. We may restrict all constructions in this section to the space $\text{LGQG}_{0,m}^{\epsilon, \theta}(Z(\theta), \beta)$. Denote by F_β the analog of the F'_β ; then there is a fibered square

$$\begin{array}{ccc} F_\beta & \xrightarrow{\quad} & F'_\beta \\ \downarrow & & \downarrow \\ \text{LGQG}_{0, m+1}^\epsilon(Z(\theta), \beta) & \longrightarrow & \text{LGQG}_{0, m+1}^\epsilon(X(\theta), \beta) \end{array}$$

Finally, we define special classes in $H_{\mathbb{C}^*}^2(\mathbb{P}^1, \mathbb{C})$. Let p_0 and p_∞ denote the pushforwards of $1 \in H_{\mathbb{C}^*}^*(\text{Spec } \mathbb{C}, \mathbb{C}) \cong \mathbb{C}[[\hbar]]$ along the equivariant inclusions $0 \hookrightarrow \mathbb{P}^1$ and $\infty \hookrightarrow \mathbb{P}^1$, respectively.

⁴As in [13], one may define the factors by convention so that this remains true for $(\beta^0, |B_0| + 1)$ and $(\beta^\infty, |B_\infty| + 1)$ unstable.

Then (choosing a \mathbb{C}^* -action on $\mathcal{O}_{\mathbb{P}^1}(1)$) we have

$$p_0|_0 = \hbar \quad p_\infty|_\infty = -\hbar \quad p_0|_\infty = p_\infty|_0 = 0.$$

6.2. The torus action on $\text{LGQ}_{0,m}^\epsilon(X(\theta), \beta)$. Torus actions on spaces of stable maps were used by Kontsevich to carry out explicit computations of Gromov-Witten invariants of toric varieties. They reduce the complicated geometry of curves in toric varieties to combinatorics of fixed point sets, which are finite and explicit. We will use the torus actions on spaces of LG-quasimaps to obtain a recursive structure, leading to the proof of Theorem 8.1. In fact, to our knowledge all of the many such “mirror” theorems in Gromov-Witten theory use torus-fixed-point localization.

For clarity, in this section we take $\theta = \theta_z^{xya}$ unless stated otherwise. For everything we do, the appropriate changes to make for the other characters will be clear.

There is a natural $T = (\mathbb{C}^*)^{13}$ action on V by scaling the coordinates. As all group actions on V that we have discussed are by scaling coordinates, they all commute. Thus we obtain T -actions on $X(\theta)$ and $[X(\theta)/\mathbb{C}_R^*]$ for each $\theta \in \Theta$. The latter induces a T -action on $\text{LGQ}_{0,m}^\epsilon(X(\theta), \beta)$, and the various bundles and maps we consider have natural T -equivariant lifts. For example, ψ_i and ev_i have natural equivariant lifts since they are defined via the geometry of maps to $[X(\theta)/\mathbb{C}_R^*]$. Similarly $\mathcal{E} = \mathcal{P} \times_{G \times \mathbb{C}_R^*} V$ has a natural lift induced by the action on V .

T -fixed locus. By a classical argument of Gromov-Witten theory, T -fixed LG-quasimaps to $X(\theta)$ are those that send C into the closure of 1-dimensional T -orbits in $[X/\mathbb{C}_R^*]$, and send all nodes, markings, and ramification points of (C, u, κ) to the T -fixed locus of $[X/\mathbb{C}_R^*]$.

We check that the T -fixed locus of $[X/\mathbb{C}_R^*]$ is where:

- $p_x = p_y = 0$,
- $z_0 = z_1 = z_2 = 0$,
- at most one of x_0, x_1 , and x_2 is nonzero, and
- at most one of y_0, y_1 , and y_2 is nonzero.

These are exactly the coordinate points of $X_R(\theta) \cong [((\mathbb{P}^2)^2 \times B\mu_3)/\mu_3] \times B\mathbb{C}_R^*$.

Similarly, the 1-dimensional T -orbits of $[X/\mathbb{C}_R^*]$ (with proper closure) are the coordinate lines in $X_R(\theta)$.

Corollary 6.6. *A T -fixed LG-quasimap (C, u, κ) to $X(\theta)$ has an associated T -fixed quasimap u^{rig} to $X(\theta)$.*

Corollary 6.7. *The T -fixed locus in $\text{LGQ}_{0,m}^\epsilon(X(\theta), \beta)$ is proper.*

As a result of the last fact, we can very closely mimic the T -localization arguments for quasimaps in [13, 7].

Definition 6.8. Write $K \cong \mathbb{C}(\lambda_1, \dots, \lambda_{13})$ for the *localized* T -equivariant cohomology of a point.

Definition 6.9. Consider a 1-dimensional T -orbit $X_{\mu,\nu}$ in $X(\theta)$ between T -fixed points μ and ν . (If such an $X_{\mu,\nu}$ exists we say μ and ν are *T -adjacent*.) We define the *tangent weight* $w(\mu, \nu)$ to be $c_1(T_\mu X_{\mu,\nu}) \in H_T^2(\mu, \mathbb{C}) \cong K$.

Remark 6.10. Everything in this section also applies to the graph space $\text{LGQG}_{0,m}^\epsilon(X(\theta), \beta)$.

7. GENERATING FUNCTIONS FOR GENUS ZERO LG-QUASIMAP INVARIANTS

Sections 7, 8 and 9 are based on Sections 5 and 7 of [13] and Section 5 of [7], respectively, with minor but necessary modifications at each step. (The techniques in [13] and [7] follow those of Givental ([23]).) We define and compare generating functions $J^{\epsilon,\theta}(t, q, \hbar)$, $S^{\epsilon,\theta}(t, q, \hbar)$, and $P^{\epsilon,\theta}(t, q, \hbar)$, encoding LG-quasimap invariants and LG-graph quasimap invariants of $Z(\theta)$. (Note that the space $\text{LGQ}_{0,m}^\epsilon(X(\theta), \beta)$ and the T -action defined in the last section do not appear in this section.) We continue to work with $\theta = \theta_z^{xya}$.

7.1. Double brackets. From now on, we fix a basis $\{\gamma_j\}$ for $\mathcal{H}(\theta)$. Let $\{\gamma^j\}$ be a dual basis with respect to the Poincaré pairing on the *nonrigidified* inertia stack $IZ(\theta)$.⁵ Let $t = \sum_j t_j \gamma_j \in \mathcal{H}(\theta)$. For $\alpha_1, \dots, \alpha_k \in \mathcal{H}(\theta)$, and $a_1, \dots, a_k \in \mathbb{Z}_{\geq 0}$, we define the *double bracket* (compare with [13, 29]):

$$(3) \quad \begin{aligned} \langle \langle \alpha_1 \psi_1^{a_1}, \dots, \alpha_k \psi_k^{a_k} \rangle \rangle_{0,k}^{\epsilon, \theta} &:= \sum_{\beta, m} \frac{q^\beta q_z^{\frac{2-(k+m)}{3}}}{m!} \langle \alpha_1 \psi_1^{a_1}, \dots, \alpha_k \psi_k^{a_k}, t, \dots, t \rangle_{0, k+m, \beta}^{\epsilon, \theta} \\ &= \sum_{\beta, m} \frac{q^{\beta - \beta_0(\theta, k+m)}}{m!} \langle \alpha_1 \psi_1^{a_1}, \dots, \alpha_k \psi_k^{a_k}, t, \dots, t \rangle_{0, k+m, \beta}^{\epsilon, \theta}. \end{aligned}$$

Here $m \geq 0$ and β runs over degrees with (β, m) θ -effective. The shifting factor $q_z^{\frac{2-(k+m)}{3}}$, which does not appear in [13], makes the double bracket an element of $\mathbb{C}[[q_x, q_y, q_z^{-1}, q_a]]$ rather than $\mathbb{C}[[q_x, q_y, q_a]]((q_z^{-1}))$. We also define *graph space double brackets* by replacing $\langle \cdot \rangle_{0, k+m, \beta}^{\epsilon, \theta}$ in (3) with $\langle \cdot \rangle_{0, k+m, \beta}^{\epsilon, \theta, Gr}$.

Notation 7.1. We write $\mathbb{C}[[q]]$ as shorthand for $\mathbb{C}[[q_x, q_y, q_z^{-1}, q_a]]$. (Analogously for $\theta \neq \theta_z^{xya}$.)

7.2. Conventions for unstable tuples. For small k , some terms of (3) correspond to unstable tuples $(\beta, k+m)$ (recall Definition 4.13). In the following sections, setting those terms to zero would not give the correct relations between generating functions. To fix this, we now *define* certain invariants corresponding to unstable tuples.

First, we motivate these conventions. We apply \mathbb{C}^* -localization to the graph space invariant

$$(4) \quad \langle \alpha_1 \psi_1^{a_1}, \dots, \alpha_m \psi_m^{a_m}, \alpha_\star \otimes p_\infty \rangle_{0, m+\star, \beta}^{\epsilon, \theta, Gr} = \int_{[\text{LGQ}_{0, m}^\epsilon(Z(\theta), \beta)]^{\text{vir}}} \prod_i \text{ev}_i^*(\alpha_i) \psi_i^{a_i} \cup \text{ev}_\star^*(\alpha_\star \otimes p_\infty).$$

The result is a sum over the fixed loci $F_{m_\infty, \beta_\infty}^{m_0, \beta_0}$. Consider the term corresponding to the locus $F_\beta = F_{\star, \beta_0(\theta, 2)}^{m, \beta}$. The tuple $(\beta_0(\theta, 2), 2)$ is unstable, which implies that \bullet is a smooth point with the marking \star . Thus by the computation in Section 6.1, if the tuple (m, β) is stable, the normal bundle to F_β is $(-\hbar^2)(\hbar - \psi_\bullet)$, under the identification of F_β with $\text{LGQ}_{0, m+\{\bullet\}}^\epsilon(Z(\theta), \beta)$. Also, $\text{ev}_\star^*(p_\infty)$ restricts on this locus to $-\hbar$ and ev_\star is identified with ev_\bullet . It follows that (4) can be written as

$$\int_{[\text{LGQ}_{0, m+\{\bullet\}}^\epsilon(Z(\theta), \beta)]^{\text{vir}}} \prod_i \text{ev}_i^*(\alpha_i) \psi_i^{a_i} \cup \frac{\text{ev}_\star^*(\alpha_\star)}{\hbar(\hbar - \psi_\bullet)} = \langle \alpha_1 \psi_1^{a_1}, \dots, \alpha_m \psi_m^{a_m}, \frac{\alpha_\star}{\hbar(\hbar - \psi_\bullet)} \rangle_{0, m+\bullet, \beta}^{\epsilon, \theta}.$$

This relation allows us to define invariants for (β, m) unstable, in the case where one entry of the bracket is of the form $\frac{\alpha}{\hbar(\hbar - \psi_i)}$. That is, we set $\langle \alpha_1 \psi_1^{a_1}, \dots, \alpha_m \psi_m^{a_m}, \frac{\alpha}{\hbar(\hbar - \psi_{m+1})} \rangle_{0, m+1, \beta}^{\epsilon, \theta}$ to be the contribution of F_β to the equivariant integral

$$\langle \alpha_1 \psi_1^{a_1}, \dots, \alpha_m \psi_m^{a_m}, \alpha \otimes p_\infty \rangle_{0, m+\star, \beta}^{\epsilon, \theta, Gr}.$$

Remark 7.2. When used in LG-quasimap invariants (rather than LG-graph quasimap invariants), we may treat \hbar as a formal variable, rather than a \mathbb{C}^* -equivariant class on \mathbb{P}^1 .

7.3. The function $J^{\epsilon, \theta}(t, q, \hbar)$. Using the conventions in the last section, we define

$$J^{\epsilon, \theta}(t, q, \hbar) := \sum_j \gamma_j \langle \langle \frac{\gamma^j}{\hbar(\hbar - \psi_1)} \rangle \rangle_{0, 1}^{\epsilon, \theta} \in \mathcal{H}(\theta)[[q]]((\hbar^{-1})).$$

⁵Using $IZ(\theta)$ instead of $\bar{IZ}(\theta)$ will make our notation much simpler. This is discussed in Section 3.1 of [7].

(This should be thought of formally as a function $\mathcal{H}(\theta) \rightarrow \mathcal{H}(\theta)[[q]]((\hbar^{-1}))$, without worrying about convergence.) The unstable tuples contributing to $J^{\epsilon, \theta}(t, q, \hbar)$ are:

$$(5) \quad \begin{aligned} (\beta, m+1) &= (\beta_0(\theta, 1), 1) & (\beta, m+1) &= (\beta_0(\theta, 2), 2) \\ (\beta, m+1) &= (\beta, 1) \text{ with } \beta_\theta < 1/\epsilon \end{aligned}$$

Calculating the terms coming from the tuples $(\beta, 1)$ with $\beta_\theta < 1/\epsilon$ (in the case $\epsilon = 0+$) is the subject of Section 9. We compute the other two terms here.

The term $(\beta_0(\theta, 1), 1)$. This term is defined, according to Section 7.2, as the F_β -contribution to the sum

$$(6) \quad \sum_j \gamma_j \langle \gamma^j \otimes p_\infty \rangle_{0, \star, \beta_0(\theta, 1)}^{\epsilon, \theta, Gr}.$$

Claim. $\text{LGQG}_{0, \star}^\epsilon(Z(\theta), \beta_0(\theta, 1))$ parametrizes the data:

- A parametrized curve $C \xrightarrow{\tau} \mathbb{P}^1$, with a marked orbifold point \star , and
- A constant map $C \rightarrow [E^2/\mu_3]$ without basepoints, and with trivial monodromy at \star .

Proof. The line bundles $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_a$, and $\mathcal{L}_{\rho_{p_z}}$ have degree zero, and thus are trivial. (We may see from Proposition 2.12 that line bundles on $\mathbb{P}_{3,1}$ have trivial monodromy at \star .) Since u lands in $[Z(\theta)/\mathbb{C}_R^*]$, the sections $\sigma_{z_0}, \sigma_{z_1}, \sigma_{z_2}, \sigma_{p_x}, \sigma_{p_y}$ are all zero. Thus up to isomorphism, (C, \mathcal{L}, σ) carries only the data of the parametrized marked curve C , the sections σ_{x_i} and σ_{y_i} , and the line bundle \mathcal{L}_z .

As $\mathcal{L}_{\rho_{p_z}}$ is trivial, we have $\mathcal{L}_z^{\otimes 3} \cong \omega_{C, \log}$. However, there is a unique such bundle up to isomorphism, with monodromy $2/3$ at \star . It has automorphism group μ_3 , acting by multiplication on fibers, which commutes with $\kappa : \mathcal{L}_z^{\otimes 3} \rightarrow \omega_{C, \log}$.

The sections σ_{x_i} and σ_{y_i} define a map $C \rightarrow [E^2/\mu_3]$. It has trivial monodromy as \mathcal{L}_a is trivial, and has no basepoints since \mathcal{L}_x and \mathcal{L}_y are trivial. \square

F_β is the locus where $\tau(\star) = \infty$, so it is isomorphic to $[E^2/\mu_3] \times B\mu_3$. (The $B\mu_3$ comes from the automorphisms of \mathcal{L}_z .) We see that F_β is a twisted sector of $IZ(\theta)$.

The virtual fundamental class is $[F_\beta]^{\text{vir}} = [F_\beta]$, and ev_\star is the μ_3 -rigidification map to a sector $[E^2/\mu_3] \subseteq \bar{I}(Z(\theta))$. The class p_∞ restricts to $-\hbar$ on F_β , and the normal bundle to $F_\beta \hookrightarrow \text{LGQG}_{0, \star}^\epsilon(Z(\theta), \beta_0(\theta, 1))$ comes from moving the image of \star on \mathbb{P}^1 , and has Euler class $-\hbar$. Thus (6) is equal to:

$$(7) \quad \sum_j \gamma_j \int_{[E^2/\mu_3] \times B\mu_3} \text{ev}_\star^*(\gamma^j) = (1 \otimes 1 \otimes 1_\zeta)_1 \int_{[E^2/\mu_3] \times B\mu_3} \text{ev}_\star^*((H_x \otimes H_y \otimes 1_{\zeta^2})_1) = 1_\theta,$$

the twisted sector of $\mathcal{H}(\theta)$ from Remark 5.9.

The term $(\beta_0(\theta, 2), 2)$. This term is the F_β -contribution to the sum

$$(8) \quad \sum_j \gamma_j \langle \gamma^j \otimes p_\infty, t \rangle_{0, \star, \beta_0(\theta, 2)}^{\epsilon, \theta, Gr}.$$

$\text{LGQG}_{0, 1+\star}^\epsilon(Z(\theta), \beta_0(\theta, 2))$ parametrizes constant maps from a parametrized curve C with *two* order 3 orbifold points to $[E^2/\mu_3]$, together with a 3rd root of $\omega_{C, \log} \cong \mathcal{O}_C$. \mathcal{L}_a may be nontrivial, and there are three choices (one trivial) for the 3rd root \mathcal{L}_z . The resulting stack is isomorphic to $IZ(\theta)$, and in particular the isomorphism is ev_\star (after composition with the rigidification $IZ(\theta) \rightarrow \bar{I}(Z(\theta))$).

The virtual fundamental class was defined to vanish on the components where \mathcal{L}_z has trivial monodromy (Remark 5.13), and on the other components it restricts to the fundamental class. The union of these components is $IZ(\theta)^{\text{nar}}$. Since $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z, \mathcal{L}_a$ have degree zero, they have opposite

monodromies at b_1 and \star , so $\text{ev}_1 = v \circ \text{ev}_1$. The normal bundle has contributions \hbar and $-\hbar$ from moving the images of \star and b_1 , respectively, on \mathbb{P}^1 . Again, $p_\infty|_{F_\beta} = -\hbar$. Thus (8) is equal to

$$(9) \quad \sum_j \gamma_j \int_{IZ(\theta)^{\text{nar}}} \frac{1}{\hbar} v^* \gamma^j \cup t = \frac{1}{\hbar} \sum_j \gamma_j \langle t, \gamma^j \rangle_{Z(\theta)} = t/\hbar.$$

Remark 7.3. In both of these calculations, the moduli spaces described parametrized maps LG-quasimaps *without basepoints*. More generally, m -marked LG-quasimaps of degree $\beta_0(\theta, m)$ never have basepoints. Thus we observe that integrals over these moduli spaces are *independent of ϵ* . In particular, the coefficient of $q^{(0,0,0,0)}$ in $J^{\epsilon,\theta}(t, q, \hbar)$ is independent of ϵ .

Proposition 7.4. $J^{\epsilon,\theta}(0, q, \hbar)$ is homogeneous of degree zero, when $\deg(q) := 0$ and $\deg(\hbar) = 2$.

Proof. By Section 7.2, any term of $J^{\epsilon,\theta}(0, q, \hbar)$ may be expressed as the F_β contribution to

$$(10) \quad \sum_j \gamma_j \langle \gamma^j \otimes p_\infty \rangle_{0,1,\beta}^{\epsilon,\theta,Gr} = (\text{ev}_1)_* (\text{ev}_1^* p_\infty) \in \mathcal{H}(\theta).$$

From Theorem 4.8, $\text{LGQG}_{0,1}(Z(\theta), \beta)$ has unshifted virtual dimension 4, so this pushforward has relative dimension 1. (This statement depends on correctly shifting degrees as in Section 5.1.) Thus (10) has degree zero. \square

Corollary 7.5. $J^{\epsilon,\theta}(t, q, \hbar) \in \mathcal{H}(\theta)[[q, \hbar^{-1}]]$.

Proof. A priori, $J^{\epsilon,\theta}(t, q, \hbar)$ may have positive powers of \hbar coming from the unstable terms. However, $J^{\epsilon,\theta}(0, q, \hbar)$ includes all unstable terms, and is homogeneous of degree zero, so this does not occur. \square

Remark 7.6. By Remark 4.16, only the tuples on the first line of (5) appear in the case $\epsilon = \infty$. In particular,

$$J^{\infty,\theta}(t, q, \hbar) = 1_\theta + \frac{t}{\hbar} + O(1/\hbar^2).$$

Combining this with Remark 7.3 implies $J^{\epsilon,\theta}(t, q, \hbar) = 1_\theta + \frac{t}{\hbar} + O(q) + O(1/\hbar^2)$, where $O(q) \in \mathcal{H}(\theta)[[q]]$ has no $q^{(0,0,0,0)}$ -coefficient.

7.4. The $S^{\epsilon,\theta}$ -operator and its inverse. We define operators on $\mathcal{H}(\theta)[[q]]((\hbar^{-1}))$ by:

$$\begin{aligned} S^{\epsilon,\theta}(t, q, \hbar)(\gamma) &= \sum_j \gamma_j \langle \langle \frac{\gamma^j}{\hbar - \psi}, \gamma \rangle \rangle_{0,2}^{\epsilon,\theta} \\ (S^{\epsilon,\theta})^*(t, q, -\hbar)(\gamma) &= \sum_j \gamma_j \langle \langle \gamma_j, \frac{\gamma}{-\hbar - \psi} \rangle \rangle_{0,2}^{\epsilon,\theta}. \end{aligned}$$

Remark 7.7. Under some conditions, we may make sense of applying these operators to power series in \hbar . The details are in Section 5.1 of [13].

Proposition 7.8. $S^{\epsilon,\theta}(t, q, \hbar) = \text{Id} + O(1/\hbar)$ and $(S^{\epsilon,\theta})^*(t, q, -\hbar) = \text{Id} + O(1/\hbar)$.

Proof. As in Corollary 7.5, the only terms of $S^{\epsilon,\theta}(t, q, \hbar)(\gamma)$ with nonnegative powers of \hbar come from unstable tuples $(\beta, m+2)$. The only such tuple is $(\beta_0(\theta, 2), 2)$. The corresponding term is the contribution of $F_\beta = F_{1,(0,0,0,0)}^{1,(0,0,0,0)}$ to

$$\sum_j \gamma_j \langle \hbar \gamma^j \otimes p_\infty, \gamma \rangle_{0,2,\beta_0(\theta,2)}^{\epsilon,\theta}$$

This contribution was calculated (Section 7.3, Equation (8)), and it is equal to γ . The argument for $(S^{\epsilon,\theta})^*(t, q, -\hbar)$ is the same. \square

Proposition 7.9.

$$(S^{\epsilon,\theta})^*(t, q, -\hbar) \left(S^{\epsilon,\theta}(t, q, \hbar)(\gamma) \right) = \gamma.$$

Proof. The series

$$\langle \langle \gamma \otimes [0], \delta \otimes [\infty] \rangle \rangle_2^{\epsilon,\theta,Gr}$$

is a sum of equivariant integrals, hence a power series in \hbar . Applying localization gives a sum over fixed components where $\tau(b_1) = 0 \in \mathbb{P}^1$ and $\tau(b_2) = \infty \in \mathbb{P}^1$. These fixed components were described in Section 6.1 as fibered products, and their normal bundles were calculated. These yield:

$$(11) \quad \langle \langle \gamma \otimes [0], \delta \otimes [\infty] \rangle \rangle_2^{\epsilon,\theta,Gr} = \sum_j \langle \langle \frac{\gamma^j}{\hbar - \psi}, \gamma \rangle \rangle_{0,2}^{\epsilon,\theta} \langle \langle \delta, \frac{\gamma_j}{-\hbar - \psi} \rangle \rangle_{0,2}^{\epsilon,\theta}.$$

The constant term in \hbar of the right side is the contribution from the fixed component $F_{1,\beta_0(\theta,2)}^{1,\beta_0(\theta,2)}$. Again, this calculation is essentially the one from Section 7.3 (Equation (8)), and the answer is $\int_{IZ(\theta)} v^* \gamma \cup \delta$.

As this is the only term of the right side of (11) with a nonnegative power of \hbar , and the left side of (11) is a power series in \hbar , we conclude:

$$\sum_j \langle \langle \frac{\gamma^j}{\hbar - \psi}, \gamma \rangle \rangle_{0,2}^{\epsilon,\theta} \langle \langle \delta, \frac{\gamma_j}{-\hbar - \psi} \rangle \rangle_{0,2}^{\epsilon,\theta} = \int_{IZ(\theta)} v^* \gamma \cup \delta.$$

Using this, we have:

$$\begin{aligned} (S^{\epsilon,\theta})^*(t, q, -\hbar) \left(S^{\epsilon,\theta}(t, q, \hbar)(\gamma) \right) &= \sum_j \gamma^j \langle \langle \gamma_j, \frac{\sum_{j'} \gamma_{j'} \langle \langle \frac{\gamma^{j'}}{\hbar - \psi}, \gamma \rangle \rangle_{0,2}^{\epsilon,\theta}}{-\hbar - \psi} \rangle \rangle_{0,2}^{\epsilon,\theta} \\ &= \sum_{j,j'} \gamma^j \langle \langle \frac{\gamma^{j'}}{\hbar - \psi}, \gamma \rangle \rangle_{0,2}^{\epsilon,\theta} \langle \langle \gamma_j, \frac{\gamma_{j'}}{-\hbar - \psi} \rangle \rangle_{0,2}^{\epsilon,\theta} \\ &= \sum_j \gamma^j \int_{IZ(\theta)} v^* \gamma \cup \gamma_j = \gamma. \end{aligned} \quad \square$$

7.5. The P -series. Finally, we define:

$$P^{\epsilon,\theta}(t, q, \hbar) := \sum_j \gamma^j \langle \langle \gamma_j \otimes p_\infty \rangle \rangle_1^{\epsilon,\theta,Gr} \in \mathcal{H}(\theta)[[q, \hbar]].$$

Proposition 7.10. $P^{\epsilon,\theta}(t, q, \hbar) = 1_\theta + O(q)$.

Proof. This coefficient of $q^{(0,0,0,0)}$ in $P^{\epsilon,\theta}(t, q, \hbar)$ is

$$(12) \quad \sum_{m,j} \frac{\gamma_j}{m!} \langle \gamma^j \otimes p_\infty, t, \dots, t \rangle_{1+m, \beta_0(\theta, m)}^{\epsilon,\theta,Gr},$$

The moduli spaces $\text{LGQG}_{0,1+m}^{\epsilon,\theta}(Z(\theta), (0, 0, \frac{-2+m}{3}, 0))$ are similar to the one described in Section 7.3, but somewhat more complicated. They have

- (1) Components corresponding to LG-quasimaps where \mathcal{L}_a is trivial, and
- (2) Components corresponding to LG-quasimaps where \mathcal{L}_a has nontrivial monodromy at some marked point.

Since \mathcal{L}_a has degree zero, these are the only possibilities.

First we consider components of type (1). As \mathcal{L}_x and \mathcal{L}_y are trivial the union of these components is isomorphic to $[E^2/\mu_3] \times \mathcal{W}_{0,1+m}(\mathbb{P}^1)$, where $\mathcal{W}_{0,1+m}(\mathbb{P}^1)$ is a moduli space of *spin curves* (see [11]). It has dimension $1 + m$ and parametrizes

- A parametrized $1 + m$ -marked curve C , and
- A 3rd root of $\omega_{C, \log}$.

Under this identification the evaluation maps ev_i are given by the product

$$(\text{id}, \text{mult}_{b_i}(\mathcal{L}_z)) : [E^2/\mu_3] \times \mathcal{W}_{0,1+m}(\mathbb{P}^1) \rightarrow [E^2/\mu_3] \times \bar{I}B\mu_3.$$

Write $\gamma_j = \gamma_{j,1} \otimes \gamma_{j,2}$, where $\gamma_{j,1} \in H^*(E^2/\mu_3, \mathbb{C})$ and $\gamma_{j,2} \in H_{CR}^*(B\mu_3)$. Similarly write $t = t_1 \otimes t_2 \in H^*(E^2/\mu_3, \mathbb{C}) \otimes H_{CR}^*(B\mu_3)$. Then (12) is equal to the product

$$(13) \quad \frac{1}{m!} \left(\int_{[E^2/\mu_3]} \gamma_{j,1} \cup t_1^m \right) \left(\int_{[\mathcal{W}_{0,1+m}(\mathbb{P}^1)]^{\text{vir}}} \text{ev}_1^*(\gamma_{j,2} \otimes p_\infty) \cup \prod_{i=1}^m \text{ev}_i^*(t_2) \right),$$

Using the projection formula, we rewrite the second integral as $\int_{\mathbb{P}^1} p_\infty \cup \alpha$, where $\alpha \in H^{1+m}(\mathbb{P}^1, \mathbb{C})$ is a nonequivariant class pushed forward from $\mathcal{W}_{0,1+m}(\mathbb{P}^1)$. Thus the second integral vanishes unless $m = 0$. The case $m = 0$ has been computed (Section 7.3, Equation (8)), and this term is equal to 1_θ .

For components of type (2), the moduli space is only a fibered product, not a product, but we similarly find that terms with $m > 0$ do not contribute. However, there are no components of type (2) with $m = 0$, since \mathcal{L}_a is a 3rd root of the trivial bundle, and when $m = 0$ there are no nontrivial 3rd roots of the trivial bundle. Thus components of type (2) do not contribute to the coefficient of $q^{(0,0,0,0)}$. \square

Factorization of $J^{\epsilon, \theta}(t, q, \hbar)$. Next we apply \mathbb{C}^* -localization to $P^{\epsilon, \theta}$. As in the proof of Proposition 7.9, we have

$$P^{\epsilon, \theta}(t, q, \hbar) = \sum_{j, j'} \gamma_j \langle \langle \frac{\gamma^{j'}}{\hbar(\hbar - \psi)} \rangle \rangle_{0,1}^{\epsilon, \theta} \langle \langle (-\hbar)\gamma_j, \frac{\gamma^{j'}}{-\hbar(-\hbar - \psi)} \rangle \rangle_{0,2}^{\epsilon, \theta}.$$

Now factoring gives

$$\begin{aligned} P^{\epsilon, \theta}(t, q, \hbar) &= \sum_j \gamma_j \langle \langle (-\hbar)\gamma_j, \frac{\sum_{j'} \gamma_{j'} \langle \langle \frac{\gamma^{j'}}{\hbar(\hbar - \psi)} \rangle \rangle_{0,1}^{\epsilon, \theta}}{-\hbar(-\hbar - \psi)} \rangle \rangle_{0,2}^{\epsilon, \theta} \\ &= \sum_j \gamma_j \langle \langle (-\hbar)\gamma_j, \frac{J^{\epsilon, \theta}(t, q, \hbar)}{-\hbar(-\hbar - \psi)} \rangle \rangle_{0,2}^{\epsilon, \theta} \\ &= \sum_j \gamma_j \langle \langle \gamma_j, \frac{J^{\epsilon, \theta}(t, q, \hbar)}{(-\hbar - \psi)} \rangle \rangle_{0,2}^{\epsilon, \theta} = (S^{\epsilon, \theta})^*(t, q, -\hbar)(J^{\epsilon, \theta}(t, q, \hbar)). \end{aligned}$$

The last expression contains no positive powers of \hbar , but $P^{\epsilon, \theta}(t, q, \hbar)$ contains no negative powers of \hbar . Thus $P^{\epsilon, \theta}(t, q, \hbar) = P^{\epsilon, \theta}(t, q) \in \mathcal{H}(\theta)[[q]]$. Applying Proposition 7.9, we have:

Corollary 7.11. $J^{\epsilon, \theta}(t, q, \hbar) = S^{\epsilon, \theta}(t, q, -\hbar)(P^{\epsilon, \theta}(t, q)).$

By Proposition 7.8 and Section 7.3, we have

$$P^{\epsilon, \theta}(t, q) = P^{\epsilon, \theta}(q) \cdot 1_\theta \in \mathbb{C}[[q]] \cdot 1_\theta$$

is independent of t and

$$J^{\epsilon, \theta}(t, q, \hbar) = P^{\epsilon, \theta}(q) \cdot 1_\theta + O(1/\hbar).$$

In particular, Remark 7.6 shows that for $\epsilon = \infty$,

$$J^{\infty, \theta}(t, q, \hbar) = 1_\theta + O(1/\hbar),$$

hence $P^{\infty, \theta}(q) = 1_\theta$. This implies

$$J^{\infty, \theta}(t, q, \hbar) = S^{\infty, \theta}(t, q, \hbar)(1_\theta),$$

which also follows from the string equation, correctly adapted as a combination of that for stable maps ([1], Theorem 8.3.1) and that appearing in FJRW theory ([20], Theorem 4.2.9).

8. MIRROR THEOREMS

8.1. Setup. In this section we will relate $J^{\infty,\theta}(t, q, \hbar)$ to $J^{\epsilon,\theta}(t, q, \hbar)$, and we describe here precisely how they are related.

In Section 7, we could have taken $t \in \mathcal{H}(\theta)[[q]]$ rather than $t \in \mathcal{H}(\theta)$ with no changes. Doing so, we may formally view $J^{\epsilon,\theta}(t, q, \hbar)$ as a map $\mathcal{H}(\theta)[[q]] \rightarrow \mathcal{H}(\theta)[[q, \hbar^{-1}]]$. It is well-known (see [33]) that the image of $J^{\infty,\theta}(t, q, \hbar)$ lies on and determines a (germ of a) cone $\mathfrak{L}_\theta \subseteq \mathcal{H}(\theta)[[q, \hbar^{-1}]]$.⁶ We will show that $J^{\epsilon,\theta}(t, q, \hbar)$ also lies on this cone also for all ϵ .

From Section 7.5, $J^{\epsilon,\theta}(t, q, \hbar)$ differs by an element of $\mathbb{C}[[q]]$ from $S^{\epsilon,\theta}(t, q, \hbar)(1_\theta)$. Thus the claim that $J^{\epsilon,\theta}(t, q, \hbar)$ lies on the cone \mathfrak{L}_θ follows from the claim that $S^{\epsilon,\theta}(t, q, \hbar)(1_\theta)$ lies on \mathfrak{L}_θ .

We will prove:

Theorem 8.1 (All-chamber mirror theorem). *There is an automorphism $\mathcal{T}^{\epsilon,\theta}$ of $\mathcal{H}(\theta)[[q]]$ such that*

$$S^{\epsilon,\theta}(t, q, \hbar)(1_\theta) = S^{\infty,\theta}(\mathcal{T}^{\epsilon,\theta}(t), q, \hbar)(1_\theta) = J^{\infty,\theta}(\mathcal{T}^{\epsilon,\theta}(t), q, \hbar).$$

In particular, the image of $S^{\epsilon,\theta}(t, q, \hbar)(1_\theta)$ is the same as the image of $J^{\infty,\theta}(t, q, \hbar)$.

To find $\mathcal{T}^{\epsilon,\theta}(t)$, we expand:

$$\begin{aligned} S^{\epsilon,\theta}(t, q, \hbar)(1_\theta) &= 1_\theta + \frac{1}{\hbar} \sum_{\substack{\beta, m, j \\ (\beta, m) \neq (\beta_0(\theta, 2), 2)}} \gamma_j \frac{q^{\beta - \beta_0(\theta, m)}}{m!} \langle \gamma^j, 1_\theta, t, \dots, t \rangle_{0, 2+m, \beta}^{\epsilon, \theta} + O(1/\hbar^2) \\ &= 1_\theta + \frac{1}{\hbar} \left(\sum_j \gamma_j \langle \langle \gamma^j, 1_\theta \rangle \rangle_{0, 2}^{\epsilon, \theta} - 1_\theta \right) + O(1/\hbar^2). \end{aligned}$$

(For consistency, the unstable part $\sum_j \gamma_j \langle \langle \gamma^j, 1_\theta \rangle \rangle_{0, 2, \beta_0(\theta, 2)}^{\epsilon, \theta}$ is defined to be 1_θ .) Set:

$$\mathcal{T}^{\epsilon,\theta}(t) := \sum_j \gamma_j \langle \langle \gamma^j, 1_\theta \rangle \rangle_{0, 2}^{\epsilon, \theta} - 1_\theta.$$

Observation 8.2. $J^{\infty,\theta}(\mathcal{T}^{\epsilon,\theta}(t), q, \hbar) = S^{\epsilon,\theta}(t, q, \hbar)(1_\theta) + O(1/\hbar^2)$.

To see that $\mathcal{T}^{\epsilon,\theta}(t)$ is an automorphism of $\mathcal{H}(\theta)[[q]]$, we equate coefficients of $q^{(0,0,0)}\hbar^{-1}$ in Corollary 7.11. Using Remark 7.6, we see that $\mathcal{T}^{\epsilon,\theta}(t) = t + O(q)$.

Our strategy for establishing the equality of $S^{\epsilon,\theta}(t, q, \hbar)(1_\theta)$ and $J^{\infty,\theta}(\mathcal{T}^{\epsilon,\theta}(t), q, \hbar)$ is to calculate both using T -localization. Since both involve integrals over $\text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)$, not $\text{LGQ}_{0,m}^\epsilon(X(\theta), \beta)$, we need to rewrite them. In particular, the coefficients of $S^{\epsilon,\theta}(t, q, \hbar)(\gamma)$ are integrals of the form

$$(14) \quad \int_{[\text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)]^{\text{vir}}} \prod_{i=1}^m \psi_i^{\alpha_i} \text{ev}_i^*(\alpha_i).$$

From Section 5.1, we only consider classes $\alpha_i = \iota^* \mathbf{a}_i$ pulled back from $\bar{I}X(\theta)$. Thus if $e : \text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta) \hookrightarrow \text{LGQ}_{0,m}^\epsilon(X(\theta), \beta)$ is the natural embedding, we may rewrite the above as

$$\int_{e_*[\text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)]^{\text{vir}}} \prod_{i=1}^m \psi_i^{\alpha_i} \text{ev}_i^*(\mathbf{a}_i).$$

⁶Here the word *cone* refers to a subset that is preserved under multiplication by elements from the “base ring” $\mathbb{C}[[q]]$. It is also a fact (which we will not need) that $\mathcal{H}(\theta)[[q]]((\hbar^{-1}))$ has a symplectic structure and that the cone is a Lagrangian submanifold.

Here we use the fact that evaluation maps and ψ classes are compatible with ι . Now we use the general fact about cosection localization that

$$(15) \quad \iota_*[\mathrm{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)]^{\mathrm{vir}} = [\mathrm{LGQ}_{0,m}^\epsilon(X(\theta), \beta)]^{\mathrm{vir}}$$

to we rewrite the S -operator (see [25]). For $\alpha \in \mathcal{H}(\theta)$, write $\tilde{\alpha}$ for the corresponding element of $H_{CR}^*(X(\theta))^{\mathrm{nar}}/(\ker \iota^*)$. Then we have

$$S^{\epsilon, \theta}(t, q, \hbar)(\gamma) = \sum_{\beta, m, j} \frac{q^{\beta - \beta_0(\theta, 2+m)}}{m!} \gamma_j \int_{[\mathrm{LGQ}_{0,2+m}^\epsilon(X(\theta), \beta)]^{\mathrm{vir}}} \frac{\mathrm{ev}_1^* \tilde{\gamma}^j}{\hbar - \psi_1} \cup \mathrm{ev}_2^* \tilde{\gamma} \cup \mathrm{ev}_3^* \tilde{t} \cdots \mathrm{ev}_{m+2}^* \tilde{t}.$$

(Note $\mathrm{LGQ}_{0,m}^\epsilon(X(\theta), \beta)$ is not proper; however, by (15) the virtual fundamental class is a homology class, in particular compactly supported, so the integral makes sense.) We also see that $S^{\epsilon, \theta}(t, q, \hbar)(\gamma)$ is pulled back from $\bar{I}X(\theta)$:

$$\tilde{S}^{\epsilon, \theta}(t, q, \hbar)(\gamma) := \sum_{\beta, m, j} \frac{q^{\beta - \beta_0(\theta, 2+m)}}{m!} \tilde{\gamma}_j \int_{[\mathrm{LGQ}_{0,2+m}^\epsilon(X(\theta), \beta)]^{\mathrm{vir}}} \frac{\mathrm{ev}_1^* \tilde{\gamma}^j}{\hbar - \psi_1} \cup \mathrm{ev}_2^* \tilde{\gamma} \cup \mathrm{ev}_3^* \tilde{t} \cdots \mathrm{ev}_{m+2}^* \tilde{t}.$$

There is no Poincaré pairing on $\bar{I}X(\theta)$, as it is not proper; however, we may still view the elements $\tilde{\gamma}_j \in H_{CR}^*(X(\theta))^{\mathrm{nar}}/(\ker \iota^*)$ as a dual basis to $\{\tilde{\gamma}^j\}$ in the following sense. Since $\bar{I}X(\theta)$ deformation retracts to $\bar{I}X_R^{\mathrm{rig}}(\theta)$, which is proper and contains $\bar{I}Z(\theta)$, there is a cohomology class \underline{Z} such that $\underline{Z} \cap X_R^{\mathrm{rig}}(\theta) = \iota_*[\bar{I}Z(\theta)] \in H_*(\bar{I}X(\theta))$. Then the Poincaré pairing on $Z(\theta)$ induces the perfect pairing on $H_{CR}^*(X(\theta))^{\mathrm{nar}}/(\ker \iota^*)$:

$$\langle \alpha, \beta' \rangle_{\underline{Z}} := \int_{[X_R^{\mathrm{rig}}(\theta)]} \alpha \cup \beta' \cup \underline{Z},$$

and under this pairing $\{\tilde{\gamma}_j\}$ and $\{\tilde{\gamma}^j\}$ are again dual bases. Alternatively, $\{\underline{Z} \cup \tilde{\gamma}_j\}$ and $\{\tilde{\gamma}^j\}$ are dual bases with respect to a perfect pairing

$$\underline{Z} \cup H_{CR}^*(X(\theta))^{\mathrm{nar}}/(\ker \iota^*) \otimes H_{CR}^*(X(\theta))^{\mathrm{nar}}/(\ker \iota^*) \rightarrow \mathbb{C}.$$

For this reason, we define

$$(16) \quad \mathfrak{Z}^{\epsilon, \theta}(t, q, \hbar)(\gamma) := \underline{Z} \cup \tilde{S}^{\epsilon, \theta}(t, q, \hbar)(\gamma).$$

It is then sufficient to show:

$$\mathfrak{Z}^{\infty, \theta}(\mathcal{T}^{\epsilon, \theta}(t), q, \hbar) = \mathfrak{Z}^{\epsilon, \theta}(t, q, \hbar)(1_\theta).$$

Remark 8.3. It will simplify things to choose lifts of classes in $\mathcal{H}(\theta)$, rather than working with elements of $H_{CR}^*(X(\theta))^{\mathrm{nar}}/(\ker \iota^*)$. Therefore in our notation we view $\tilde{\gamma}_j$, $\tilde{\gamma}^j$, $\tilde{\gamma}$, \tilde{t} , and $\mathfrak{Z}^{\epsilon, \theta}(t, q, \hbar)(\gamma)$ as elements of $H_{CR}^*(X(\theta))^{\mathrm{nar}}[[q, \hbar^{-1}]]$. Of course, they are not well-defined, but their pullbacks to $Z(\theta)$ are.

We are now in a situation to follow the arguments of [13]. For each fixed point μ of $\bar{I}X(\theta)$, consider the T -equivariant integral

$$\mathfrak{Z}_\mu^{\epsilon, \theta}(t, q, \hbar)(\gamma) := i_\mu^* \mathfrak{Z}^{\epsilon, \theta}(t, q, \hbar)(\gamma) = \int_{[X_R^{\mathrm{rig}}(\theta)]} \delta_\mu \cup \mathfrak{Z}^{\epsilon, \theta}(t, q, \hbar)(\gamma),$$

where $\delta_\mu \in H_{CR, T, \mathrm{loc}}^*(X(\theta))$ is the equivariant fundamental class of μ and $i_\mu : \mu \hookrightarrow \bar{I}X(\theta)$ is the inclusion. Precisely, as $\bar{I}X(\theta)$ has isolated fixed points, the Atiyah-Bott localization formula states that its localized equivariant cohomology groups $H_{CR, T, \mathrm{loc}}^*(X(\theta))$ are generated by pushforwards

of fundamental classes of the fixed points from the equivariant cohomology of the fixed locus. We rewrite

$$(17) \quad \mathfrak{Z}_\mu^{\epsilon, \theta}(t, q, \hbar)(\gamma) = \sum_{m, \beta} \frac{q^{\beta - \beta_0(\theta, 2+m)}}{m!} \int_{[\text{LGQ}_{0, 2+m}^\epsilon(X(\theta), \beta)]^{\text{vir}}} \frac{\text{ev}_1^* \delta_\mu}{\hbar - \psi_1} \cup \text{ev}_2^* \tilde{\gamma} \cup \text{ev}_3^* \tilde{t} \cup \dots \cup \text{ev}_{m+2}^* \tilde{t}.$$

These are the objects of interest in the next section.

8.2. Localization and recursion. From Section 6.2, (17) has a natural equivariant lift, so we apply the fixed-point localization formula to write:

$$(18) \quad \mathfrak{Z}_\mu^{\epsilon, \theta}(t, q, \hbar)(\gamma) = \sum_{m, \beta, F} \frac{q^{\beta - \beta_0(\theta, 2+m)}}{m!} \int_{[F]^{\text{vir}}} \frac{i_F^* \left(\frac{\text{ev}_1^* \delta_\mu}{\hbar - \psi_1} \cup \text{ev}_2^* \tilde{\gamma} \cup \dots \cup \text{ev}_{m+2}^* \tilde{t} \right)}{e(N_F^{\text{vir}})},$$

where $i_F : F \hookrightarrow \text{LGQ}_{0, 2+m}^\epsilon(X(\theta), \beta)$ is the inclusion of a component of the fixed locus, $[F]^{\text{vir}}$ is the virtual fundamental class from the T -fixed part of $R^\bullet \pi_* \mathcal{E}$, and N_F^{vir} is the T -equivariant virtual normal bundle, defined to be the T -moving part of $R^\bullet \pi_* \mathcal{E}$.

We recall terminology from [13].

Definition 8.4. For each fixed point μ of $\bar{I}X(\theta)$, we partition the components of the T -fixed locus of $\text{LGQ}_{0, 2+m}^\epsilon(X(\theta), \beta)$ into three subsets:

- $V(\mu, \beta, 2+m)$ consists of components for which the first marking does not map to μ ,
- $\text{In}(\mu, \beta, 2+m)$ consists of components for which the first marking maps to μ and is on a contracted component of C (see Definition 4.17), and
- $\text{Rec}(\mu, \beta, 2+m)$ consists of components for which the first marking maps to μ and is not on a contracted component of C . In this case, u^{rig} sends this component to a fixed curve in $\bar{I}X(\theta)$ connecting μ to a unique other fixed point, which we denote by ν .

This is Lemma 7.5.1 of [13]:

Lemma 8.5. For each $(\beta, (k_j)_j)$ with $\sum_j k_j = m$ the coefficient of $q^{\beta - \beta_0(\theta, 2+m)} \prod_j t_j^{k_j}$ in $\mathfrak{Z}_\mu^{\epsilon, \theta}(\gamma)$ is a rational function of \hbar with coefficients in K (see Section 6.2). This rational function decomposes as a finite sum of rational functions with denominators either powers of \hbar , of powers of linear factors $\hbar - \alpha$, where $-\alpha$ is one of the weights of the T -representation $T_\mu(\bar{I}X(\theta))$ for some $n \in \mathbb{Z}_{>0}$.

Proof. The proof in [13] requires essentially no modification, and we summarize it here.

From (18), the coefficient of $q^{\beta - \beta_0(\theta, 2+m)} \prod_j t_j^{k_j}$ in $\mathfrak{Z}_\mu^{\epsilon, \theta}(t, q, \hbar)(\gamma)$ is:

$$(19) \quad \sum_F \frac{1}{(m!)} \int_{[F]^{\text{vir}}} \frac{i_F^* \left(\frac{\text{ev}_1^* \delta_u}{\hbar - \psi} \cup A \right)}{e(N_F^{\text{vir}})} = \sum_{F, a} \frac{1}{(m! \hbar^{a+1})} \int_{[F]^{\text{vir}}} \frac{i_F^* (\psi_1^a \text{ev}_1^* \delta_u \cup A)}{e(N_F^{\text{vir}})},$$

where A is the product of factors from the evaluation maps $2, \dots, 2+m$, and depends on β and $(k_j)_j$.

- On components in $V(\mu, \beta, 2+m)$, the factor $\text{ev}_1^* \delta_\mu$ restricts to zero.
- On components in $\text{In}(\mu, \beta, 2+m)$, ψ_1 is nonequivariant, hence nilpotent, so the denominators are (bounded) powers of \hbar .
- On components in $\text{Rec}(\mu, \beta, 2+m)$, the ψ_1 is an equivariant class. However, if d is the degree of u^{rig} on the component containing b_1 , then the fibers of the $(T_{b_1}^* C)^{\otimes d}$ is naturally isomorphic to $T_\mu^* X_{\mu, \nu}$ from Section 6.2. Thus the left side of (19) has a simple pole at $\psi_1 = \frac{-w(\mu, \nu)}{d}$. \square

This is Lemma 7.5.2 of [13], and is essentially unchanged from Proposition 4.4 of [23].

Lemma 8.6. $\mathfrak{Z}_\mu^{\epsilon, \theta}(t, q, \hbar)$ satisfies the recursion

$$(20) \quad \mathfrak{Z}_\mu^{\epsilon, \theta}(t, q, \hbar) = R_\mu^{\epsilon, \theta}(t, q, \hbar) + \sum_{\substack{\nu \text{ } T\text{-adjacent} \\ \text{to } \mu}} \sum_{d=1}^{\infty} q^{d\beta(\mu, \nu)} \frac{C_{\mu, \nu, d}}{\hbar + \frac{w(\mu, \nu)}{d}} \mathfrak{Z}_\nu^{\epsilon, \theta} \left(t, q, \frac{w(\mu, \nu)}{d} \right),$$

such that

- $R_\mu^{\epsilon, \theta}(t, q, \hbar)$ is a power series in $1/\hbar$ such that for each $(\beta, (k_j)_j)$ with $\sum k_j = m$, the coefficient of $q^{\beta - \beta_0(\theta, 2+m)} \prod_j t_j^{k_j}$ is a polynomial in $1/\hbar$,
- $\beta(\mu, \nu)$ is a degree dependent only on μ and ν ,
- $w(\mu, \nu)$ is the tangent weight defined in Section 6.2, and
- $C_{\mu, \nu, d}$ is independent of ϵ .

The proof is similar to that in [13]. Note also that in the case $\theta = \theta_{xyz}^a$ the second term is zero, as every component is contracted.

Proof. $\mathfrak{Z}_\mu^{\epsilon, \theta}(t, q, \hbar)$ has a single unstable term, from the unstable tuple $(\beta_0(\theta, 2), 2)$. Using Section 7.4, the contribution is $\langle [Z(\theta)] \cup \tilde{\gamma}, \delta_\mu \rangle$.

We analyze the contributions from $V(\mu, \beta, 2+m)$, $\text{In}(\mu, \beta, 2+m)$, and $\text{Rec}(\mu, \beta, 2+m)$ to (18). As in Lemma, 8.5, the contribution of components in $V(\mu, \beta, 2+m)$, is zero.

Consider a fixed component in $\text{In}(\mu, \beta, 2+m)$. This parametrizes LG-quasimaps such that the associated quasimap $C \rightarrow [E^2/\mu_3]$ sends b_1 to μ and contracts the component containing b_1 . As in the proof of Lemma 8.5, the contribution is a power series in $1/\hbar$, whose $q^{\beta - \beta_0(\theta, 2+m)} \prod_j t_j^{k_j}$ -coefficient is an element of $K[1/\hbar]$. We define $R_\mu^{\epsilon, \theta}(t, q, \hbar)$ to be the sum of contributions from components in $\text{In}(\mu, \beta, 2+m)$.

We now consider a fixed component $M \in \text{Rec}(\mu, \beta, 2+m)$, corresponding to the term of (18):

$$(21) \quad \frac{q^{\beta - \beta_0(\theta, 2+m)}}{m!} \int_{[M]^{\text{vir}}} \frac{\frac{\text{ev}_1^* \delta_\mu}{\hbar - \psi_1} \cup \text{ev}_2^* \tilde{\gamma} \cup \dots \cup \text{ev}_{m+2}^* \tilde{t}}{e(N_M^{\text{vir}})}.$$

Let ν be as in Definition 8.4. LG-quasimaps in $(\text{As with } \mu, \nu \text{ naturally lives in } \bar{I}X(\theta)^T \text{ rather than } X(\theta)^T.)$ By gluing LG-quasimaps, we can write M as a fibered product $M' \times_{\bar{I}X(\theta)} M''$, where M' is a T -fixed component of $\text{LGQ}_{0,1+\bullet}^\epsilon(X(\theta), \beta')$ and M'' is a T -fixed component of $\text{LGQ}_{0,m+1+\bullet}^\epsilon(X(\theta), \beta - \beta')$. (Note that the meaning of \bullet differs very slightly from that in Section 6.1.) Here β' is the degree of (C', u', κ') . The maps to $\bar{I}X(\theta)$ are, in the first case, the evaluation map at \bullet , and in the second case, the evaluation map at \bullet , composed with the inversion map on $\bar{I}Z(\theta)$. (See Sections 2.2 and 2.3.)

As C' has a single marked point, a single node, and no basepoints, we have $\beta'_z = 0$. Similarly $\beta'_a = 0$. Also, by the characterization of 1-dimensional T -orbits in Section 6.2, either $\beta'_x = 0$ or $\beta'_y = 0$, and by the noncontractedness of C' , the other is a positive integer. Thus it is of the form $d\beta(\mu, \nu)$, where $\beta(\mu, \nu)$ is either $(1, 0, 0, 0)$ or $(0, 1, 0, 0)$. In particular, $\beta' - \beta_0(\theta, 2) \neq (0, 0, 0, 0)$.

We wish to write (21) as a product of integrals over M' and M'' . To do this, we need to compute the virtual class $[M]^{\text{vir}}$ in terms of $[M']^{\text{vir}}$ and $[M'']^{\text{vir}}$. Smoothing the node $o : C' \cap C''$ gives the distinguished triangle of relative perfect obstruction theories:

$$R^\bullet \pi_* \mathcal{E} \rightarrow R^\bullet \pi_* \mathcal{E}|_{C'} \oplus R^\bullet \pi_* \mathcal{E}|_{C''} \rightarrow R^\bullet \pi_* \mathcal{E}|_o \rightarrow R^\bullet \pi_* \mathcal{E}[1].$$

The term $R^\bullet \pi_* \mathcal{E}|_o$ is isomorphic as a G -bundle over M to the trivial bundle with fiber V , concentrated in degree zero.

We instead need a perfect obstruction theory relative to the stack $\mathfrak{M}_{0,m}^{\text{tw}}$ (Section 2.2). The difference comes from the relative tangent complex $\mathbb{T}_{\mathfrak{M}_{0,m}^{\text{tw}}}$. This is equal to $\mathcal{P} \times_{G \times \mathbb{C}_R^*} \mathfrak{g}$, concentrated

in degree -1, where \mathfrak{g} is the Lie algebra of G . Thus if we denote by $\mathfrak{F}^\bullet(C)$ the perfect obstruction theory of $\mathrm{LGQ}_{0,m}^\epsilon(X(\theta), \beta)$ relative to $\mathfrak{M}_{0,m}^{\mathrm{tw}}$, and by $\mathfrak{F}^\bullet(C')$, etc., the corresponding perfect obstruction theories on M' , etc., the triangle above becomes

$$\mathfrak{F}^\bullet(C) \rightarrow \mathfrak{F}^\bullet(C') \oplus \mathfrak{F}^\bullet(C'') \rightarrow \mathfrak{F}^\bullet(o) \rightarrow \mathfrak{F}^\bullet[1],$$

where every fiber of $\mathfrak{F}(o)$ can be canonically identified with $T_\nu \bar{I}X(\theta)$. As the T -fixed points of $\bar{I}X(\theta)$ are isolated, $T_\nu \bar{I}X(\theta)$ has no T -fixed part.

Now, to be able to make statements about the *absolute* obstruction theory of $\mathrm{LGQ}_{0,m}^\epsilon(X(\theta), \beta)$, we need to analyze the tangent complex of $\mathfrak{M}_{0,m}^{\mathrm{tw}}$. Again we have a triangle

$$\mathbb{T}_{\mathfrak{M}_{0,m}^{\mathrm{tw}}}(C) \rightarrow \mathbb{T}_{\mathfrak{M}_{0,m}^{\mathrm{tw}}}(C') \oplus \mathbb{T}_{\mathfrak{M}_{0,m}^{\mathrm{tw}}}(C'') \rightarrow \mathbb{T}_{\mathfrak{M}_{0,m}^{\mathrm{tw}}}(o) \rightarrow \mathbb{T}_{\mathfrak{M}_{0,m}^{\mathrm{tw}}}[1],$$

and here $\mathbb{T}_{\mathfrak{M}_{0,m}^{\mathrm{tw}}}(o)$ is the deformation space of the node o , with each fiber canonically isomorphic to $T_o C' \otimes T_o C''$. The factor $T_o C'$ gives a topologically trivial bundle over M up to torsion, with T -weight $w(\nu, \mu)/d$. The factor $T_o C''$ may be topologically nontrivial (depending on M''), but in any case the T -action on $\mathbb{T}_{\mathfrak{M}_{0,m}^{\mathrm{tw}}}(o)$ is nontrivial. Thus we have, exactly as in [13]:

$$\begin{aligned} [M]^{\mathrm{vir}} &= [M']^{\mathrm{vir}} \times [M'']^{\mathrm{vir}} \\ \frac{1}{e^T(N_M^{\mathrm{vir}})} &= \frac{e^T(T_\nu \bar{I}X(\theta))}{e^T(N_{M'}^{\mathrm{vir}}) e^T(N_{M''}^{\mathrm{vir}}) \left(\frac{w(\nu, \mu)}{d} - \psi_1^{M''} \right)}. \end{aligned}$$

Since $\mathrm{ev}_1|_{M''}$ is a constant map to $\nu \in \bar{I}X(\theta)$, and $w(\nu, \mu) = -w(\mu, \nu)$, (21) is equal to the product:

$$\left(q^{\beta' - \beta_0(\theta, 2)} \int_{[M']^{\mathrm{vir}}} \frac{\mathrm{ev}_1^* \delta_\mu}{\left(\hbar + \frac{w(\mu, \nu)}{d} \right) e^T(N_{M'}^{\mathrm{vir}})} \right) \left(\frac{q^{(\beta - \beta') - \beta_0(\theta, m+2)}}{m!} \int_{[M'']^{\mathrm{vir}}} \frac{\mathrm{ev}_1^*(\delta_\nu) \cup \mathrm{ev}_2^*(\tilde{\gamma}) \cup \prod_i \mathrm{ev}_i^*(t)}{\left(\frac{-w(\mu, \nu)}{d} - \psi_1 \right) e^T(N_{M''}^{\mathrm{vir}})} \right)$$

The factor $\hbar + \frac{w(\mu, \nu)}{d}$ is pulled back from $H_{T \times \mathbb{C}^*}^*(\mathrm{Spec} \mathbb{C}, \mathbb{C})$, so may be factored out. The resulting integral $C_{\mu, \nu, d}$ is over a moduli space of sections with *no basepoints*, hence it is independent of ϵ . Summing over M , β , and m , we get (20). \square

Lemma 8.7. Define $(qe^{L_\rho})^\beta = q^\beta e^{\beta \rho}$. Then for any $\gamma \in \mathcal{H}(\theta)[[q]]$, the expression

$$D(3_\mu^{\epsilon, \theta}) := \left(3_\mu^{\epsilon, \theta}(t, qe^{YzL_\vartheta}, \hbar)(\gamma) \right) \left(3_{v(\mu)}^{\epsilon, \theta}(t, q, -\hbar)(\gamma) \right)$$

is an element of $K[[q, Y, \hbar]]$.

Proof. There is an important line bundle $U(\mathcal{L})_\theta^{\beta_1, \beta_2}$ on each quasimap graph space, defined in [13]. We define a modified version for LG-quasimap graph spaces.

The line bundle L_ϑ on $[V/(G \times \mathbb{C}_R^*)]$ induces an embedding $[V/(G \times \mathbb{C}_R^*)] \hookrightarrow [\mathbb{C}^{N+1}/\mathbb{C}^*]$. Given an LG-graph quasimap (C, u, κ, τ) of degree β , composing gives a prestable graph quasimap $C \rightarrow \mathbb{P}^N$ of degree β_ϑ . (The fact that this quasimap is prestable in the sense of [13] comes from the fact that $V^{ss}(\vartheta) = V^{ss}(\theta)$.) Note that $\mathcal{L}_\vartheta = u^* L_\vartheta$ has trivial monodromy at every marked point of C , and indeed the map $C \rightarrow \mathbb{P}^N$ factors through the coarse moduli space \bar{C} of C by the definition of \bar{C} . Done in families, this construction yields a map $\mathrm{LGQG}_{0,m}^\epsilon(X(\theta), \beta) \mathcal{Q}(\beta_\vartheta)$, where $\mathcal{Q}(\beta_\vartheta)$ is a stack of prestable *nonorbifold* graph quasimaps to \mathbb{P}^N of degree β_ϑ .

The stack $\mathcal{Q}(\beta_\vartheta)$ has a forget-and-contract map to a stack $\mathcal{Q}'(\beta_\vartheta)$ as in Section 3 of [13], remembering only the restriction of a quasimap to the parametrized component; all marked points are forgotten (possible since the orbifold structure has been removed), and nodes are replaced with basepoints of degree equal to the total degree of the line bundle “on the other side” of the node. The stack $\mathcal{Q}'(\beta_\vartheta)$ parametrizes sections of line bundles on \mathbb{P}^1 , with no stability conditions — in fact, it is a projective space. Denote by $U(\mathcal{L})_\vartheta$ the pullback to $\mathrm{LGQG}_{0,m}^\epsilon(X(\theta), \beta)$ of $\mathcal{O}_{\mathcal{Q}'(\beta_\vartheta)}(1)$.

Instead of forgetting the marked points, one may replace them with basepoints. Fix degrees $\beta(1)$ and $\beta(2)$. Write $\beta_{\vartheta}(1)$ and $\beta_{\vartheta}(2)$ for the corresponding integers as in Definition 4.5. Then there is a map

$$\Phi : \text{LGQG}_{0,m}^{\epsilon}(X(\theta), \beta) \rightarrow \mathcal{Q}'(\beta_{\vartheta} + \beta(1)_{\vartheta} + \beta(2)_{\vartheta}),$$

which as above sends (C, u, κ) to a quasimap $\mathbb{P}^1 \rightarrow \mathbb{P}^N$, with “artificial” basepoints added at $\tau(b_1)$ and $\tau(b_2)$. We define

$$U(\mathcal{L})_{\vartheta}^{\beta(1), \beta(2)} := \Phi^* \mathcal{O}_{\mathcal{Q}'(\beta_{\vartheta} + \beta(1)_{\vartheta} + \beta(2)_{\vartheta})}(1).$$

Let $F_{\mu}(2+m, \beta) \subseteq \text{LGQG}_{0,2+m}^{\epsilon}(X(\theta), \beta)^T$ be the open and closed substack of T -fixed (but not necessarily \mathbb{C}^* -fixed) LG-quasimaps for which the parametrized component is contracted to μ . It is \mathbb{C}^* -invariant but not \mathbb{C}^* -fixed, as there may be basepoints, nodes, and marked points mapped by τ to $\mathbb{P}^1 \setminus \{0, \infty\}$. Write

$$\gamma = \sum_{\beta} q^{\beta} \gamma_{\beta} \in \mathcal{H}(\theta)[[q]]$$

and consider the series of T -equivariant integrals:

$$(22) \quad \sum_{m, \beta} \frac{q^{\beta - \beta_0(\theta, 2+m)}}{m!} \sum_{\beta(1), \beta(2)} q^{\beta(1)} q^{\beta(2)} \int_{[F_{\mu}(2+m, \beta)]^{\text{vir}}} \frac{e^{c_1(U(\mathcal{L})_{\vartheta}^{\beta(1), \beta(2)})^Y} \text{ev}_1^*(\tilde{\gamma}_{\beta(1)} \otimes p_0) \text{ev}_2^*(\tilde{\gamma}_{\beta(2)} \otimes p_{\infty}) \prod_{i=3}^{2+m} \text{ev}_i^*(t)}{e^{T(N_{F_{\mu}(2+m, \beta)}^{\text{vir}})}} \in K[[\hbar]].$$

Since the denominator is a class in the T -equivariant cohomology of $F_{\mu}(2+m, \beta)$, it does not contain \hbar .

We apply \mathbb{C}^* -localization to compute the integral. The contribution from a fixed component $F_{B_0, \beta^0, \mu}^{B_{\infty}, \beta^{\infty}} := F_{B_0, \beta^0}^{B_{\infty}, \beta^{\infty}} \cap F_{\mu}(2+m, \beta)$ is zero unless $b_1 \in B_0$ and $b_2 \in B_{\infty}$. In this case, since we have seen that $e(N_{F_{B_0, \beta^0, \mu}^{B_{\infty}, \beta^{\infty}}|_{F_{\mu}(2+m, \beta)}}^{\text{vir}}) = (-\hbar^2)(\hbar - \psi_{\bullet})(-\hbar - \psi_{\bullet})$, we get the integral:

$$(23) \quad \int_{[F_{B_0, \beta^0}^{B_{\infty}, \beta^{\infty}} \cap F_{\mu}(2+m, \beta)]^{\text{vir}}} \frac{e^{c_1(U(\mathcal{L})_{\vartheta}^{\beta(1), \beta(2)})^Y} \text{ev}_1^*(\tilde{\gamma}_{\beta(1)}) \text{ev}_2^*(\tilde{\gamma}_{\beta(2)}) \prod_{i=3}^{2+m} \text{ev}_i^*(t)}{e^{T(N_{F_{\mu}(2+m, \beta)}^{\text{vir}})} (\hbar - \psi_{\bullet})(-\hbar - \psi_{\bullet})}.$$

As before we may write $F_{B_0, \beta^0, \mu}^{B_{\infty}, \beta^{\infty}}$ as a (not fibered) product $M_0 \times \widehat{M} \times M_{\infty}$, where

$$\begin{aligned} M_0 &= (\text{LGQ}_{0,|B_0|+\bullet}^{\epsilon}(X(\theta), \beta^0))_{\mu}^T \\ \widehat{M} &= (\text{LGQG}_{0,2}^{\epsilon}(X(\theta), \beta_0(\theta, 2)))_{\mu}^{T \times \mathbb{C}^*} \\ M_{\infty} &= (\text{LGQ}_{0,|B_{\infty}|+\bullet}^{\epsilon}(X(\theta), \beta^{\infty}))_{\mu}^T. \end{aligned}$$

Here the superscript μ refers only to components where the extra marked point (or, for \widehat{M} , the entire curve C) is mapped to μ . \widehat{M} is a union of points, each corresponding to choices of monodromies.

As in Lemma 8.6, we write (23) as a product of integrals over M_0 and M_{∞} . Smoothing the nodes \bullet and \bullet shows

$$N_{F_{\mu}(2+m, \beta)}^{\text{vir}} = N_{M_0}^{\text{vir}} \oplus N_{M_{\infty}}^{\text{vir}},$$

where the normal bundles on the right are taken relative to the ambient spaces $\text{LGQ}_{0,|B_0|+\bullet}^{\epsilon}(X(\theta), \beta^0)$ and $\text{LGQ}_{0,|B_{\infty}|+\bullet}^{\epsilon}(X(\theta), \beta^{\infty})$. The line bundle $U(\mathcal{L})_{\vartheta}^{\beta(1), \beta(2)}$ can be expressed on the product $M_0 \times \widehat{M} \times M_{\infty}$ as follows. As the construction above involves restricting to the parametrized component, the

map $M_0 \times \widehat{M} \times M_\infty \rightarrow \mathcal{Q}'(\beta_\vartheta + \beta(1)_\vartheta + \beta(2)_\vartheta)$ is constant, so the restriction of $U(\mathcal{L})_{\vartheta}^{\beta(1), \beta(2)}|_{M_0 \times \widehat{M} \times M_\infty}$ is topologically trivial.

We compute the $(T \times \mathbb{C}^*)$ -weight as follows. Let $(C, u, \kappa) \in M_0 \times \widehat{M} \times M_\infty$, with $C \cong C_0 \cup \widehat{C} \cup C_\infty$. Then $\Phi(C, u, \kappa)$ is a quasimap $\mathbb{P}^1 \rightarrow \mathbb{P}^N$, given in coordinates by

$$[s : t] \mapsto [a_0 s^{\beta_\vartheta^0 + \beta(1)_\vartheta} t^{\beta_\vartheta^\infty + \beta(2)_\vartheta} : \dots : a_N s^{\beta_\vartheta^0 + \beta(1)_\vartheta} t^{\beta_\vartheta^\infty + \beta(2)_\vartheta}].$$

Here the a_i s are determined by μ . The weight of $U(\mathcal{L})_{\vartheta}^{\beta(1), \beta(2)}$ at (C, u, κ) is equal to the weight of $\mathcal{O}_{\mathcal{Q}'(\beta_\vartheta + \beta(1)_\vartheta + \beta(2)_\vartheta)}(1)$ at $\Phi(C, u, \kappa)$. By the definition of the map Φ , this is the T -weight of L_ϑ at μ , denoted $w_{\mu, \vartheta}$. From the choice of coordinates in Section 6.1, the \mathbb{C}^* -weight is $(\beta_\vartheta^0 + \beta(1)_\vartheta)\hbar$.

Now we may factor the integral (23) as:

$$e^{(w_{\mu, \vartheta})Y} \left(\int_{[(\text{LGQ}_{0, |B_0| + \bullet}^\epsilon(X(\theta), \beta^0))_\mu^T]^{\text{vir}}} \frac{e^{(\beta_\vartheta^0 + \beta_\vartheta(1))\hbar Y} \text{ev}_\bullet(\delta_\mu) \text{ev}_1(\tilde{\gamma}_{\beta(1)}) \prod_{i=2}^{|B_0|} \text{ev}_i^*(t)}{e^{T(N_{(\text{LGQ}_{0, |B_0| + \bullet}^\epsilon(X(\theta), \beta^0))_\mu^T}^{\text{vir}})(\hbar - \psi_\bullet)}} \right) \\ \cdot \left(\int_{[(\text{LGQ}_{0, |B_\infty| + \bullet}^\epsilon(X(\theta), \beta^\infty))_{v(\mu)}^T]^{\text{vir}}} \frac{\text{ev}_\bullet(v^* \delta_\mu) \text{ev}_1(\tilde{\gamma}_{\beta(2)}) \prod_{i=2}^{|B_\infty|} \text{ev}_i^*(t)}{e^{T(N_{(\text{LGQ}_{0, |B_\infty| + \bullet}^\epsilon(X(\theta), \beta^\infty))_{v(\mu)}^T}^{\text{vir}})(-\hbar - \psi_\bullet)}} \right).$$

For compactness, we write this as $e^{(w_{\mu, \vartheta})Y} \mathcal{S}(|B_0|) \mathcal{S}(|B_\infty|)$. Summing gives:

$$e^{(w_{\mu, \vartheta})Y} \sum_{\substack{B_0, B_\infty, \\ \beta^0, \beta^\infty, \\ \beta(1), \beta(2)}} \frac{q^{\beta^0 + \beta^\infty + \beta(1) + \beta(2) - \beta_0(\theta, |B_0| + |B_\infty|)}}{(|B_0| + |B_\infty|)!} \mathcal{S}(|B_0|) \mathcal{S}(|B_\infty|) \\ = e^{(w_{\mu, \vartheta})Y} \sum_{\substack{m_0, m_\infty, \\ \beta^0, \beta^\infty, \\ \beta(1), \beta(2)}} \frac{q^{\beta^0 + \beta^\infty + \beta(1) + \beta(2) - \beta_0(\theta, m_0 + m_\infty)}}{m_0! m_\infty!} \mathcal{S}(m_0) \mathcal{S}(m_\infty) \\ = e^{(w_{\mu, \vartheta})Y} \left(\sum_{m_0, \beta^0, \beta(1)} \frac{q^{\beta^0 + \beta(1) - \beta_0(\theta, m_0 + 1)}}{m_0!} \mathcal{S}(m_0) \right) \left(\sum_{m_\infty, \beta^\infty, \beta(2)} \frac{q^{\beta^\infty + \beta(2) - \beta_0(\theta, m_\infty + 1)}}{m_\infty!} \mathcal{S}(m_\infty) \right) \\ = e^{(w_{\mu, \vartheta})Y} \left(\sum_{\beta(1)} (q e^{Y \hbar L_\vartheta})^{\beta(1)} \mathfrak{Z}_\mu^{\epsilon, \theta}(t, q e^{Y \hbar L_\vartheta}, \hbar)(\gamma_{\beta(1)}) \right) \left(\sum_{\beta(2)} q^{\beta(2)} \mathfrak{Z}_{v(\mu)}^{\epsilon, \theta}(t, q, -\hbar)(\gamma_{\beta(2)}) \right) \\ = e^{(w_{\mu, \vartheta})Y} \left(\mathfrak{Z}_\mu^{\epsilon, \theta}(t, q e^{Y \hbar L_\vartheta}, \hbar)(\gamma) \right) \left(\mathfrak{Z}_{v(\mu)}^{\epsilon, \theta}(t, q, -\hbar)(\gamma) \right). \quad \square$$

We have now assembled all of the necessary pieces to prove our mirror theorem.

Proof of Theorem 8.1. The theorem now follows from Uniqueness Lemma 7.7.1 of [13], applied to the systems:

$$\{\mathfrak{Z}_\mu^{\epsilon, \theta}(t, q, \hbar)(1_\theta), \mu \in \overline{IX}(\theta)^T\} \\ \{\mathfrak{Z}_\mu^{\infty, \theta}(\mathcal{T}^{\epsilon, \theta}(t), q, \hbar)(1_\theta), \mu \in \overline{IX}(\theta)^T\}.$$

(As in Section 3.7.3, Item (3) of [7], we modify Condition (5) of the Uniqueness Lemma slightly.) In particular, the Uniqueness Lemma requires five properties to hold, and they are verified in:

- (1) Lemma 8.5,
- (2) Lemma 8.6,
- (3) Lemma 8.7,

- (4) Observation 8.2, and
 (5) Remark 7.3. □

9. CALCULATING THE I -FUNCTIONS

In this section, we compute $I^\theta(q, \hbar) := J^{0+, \theta}(0, q, \hbar)$ for any $\theta \in \Theta$. The following three observations allow explicit computations.

Observation 9.1. $\text{LGQG}_{0,1}^{0+}(X(\theta), \beta)$ is proper. To see this, consider $(C, \mathcal{L}, \sigma) = (C, u, \kappa) \in \text{LGQG}_{0,1}^{0+}(X(\theta), \beta)$. If x is a superscript variable, then $\beta_x \geq 0$. Hence the bundle $\mathcal{L}_{p_x} \cong \mathcal{L}_x^{-3} \otimes \omega_{C, \log}$ has negative degree, since $\deg \omega_{C, \log} = -2 + 1 = -1 < 0$. If x is a subscript variable, we saw in Proposition 4.21 that \mathcal{L}_x and $\mathcal{L}_x \otimes \mathcal{L}_a^*$ have no global sections. From these, properness follows by a standard argument.

Alternatively, one may show that $\text{LGQG}_{0,1}^{0+}(X(\theta), \beta)$ is isomorphic to $\text{LGQG}_{0,1}^{0+}(Z'(\theta), \beta)$, where $Z'(\theta) \cong [(\mathbb{P}^2)^2 \times B\mu_3]/\mu_3$ is the critical locus of a polynomial W' inside a quotient $X'(\theta')$. Then Theorem 4.8 asserts that $\text{LGQG}_{0,1}^{0+}(Z'(\theta), \beta)$ is proper.

Observation 9.2. The universal curve U'_β over the distinguished fixed part F'_β (see Definition 6.4) is trivial, with fibers canonically isomorphic to $\mathbb{P}_{3,1}$, as follows. Recall that F'_β parametrizes LG-quasimaps (C, u, κ) where C has a single marking b_1 with $\tau(b_1) = \infty \in \mathbb{P}^1$. Further, the degree β is concentrated at $\tau^{-1}(0)$. The ϵ -stability condition implies that $\tau^{-1}(0)$ is a single point, and u has a basepoint of degree β there. All fibers are canonically identified with $\mathbb{P}_{3,1}$, so $U'_\beta \rightarrow F'_\beta$ is a trivial family. We write ϖ for the projection $U'_\beta \rightarrow \mathbb{P}_{3,1}$.

Observation 9.3. For each summand \mathcal{L}_ρ of \mathcal{E} , at least one of $H^0(C, \mathcal{L}_\rho)$ and $H^1(C, \mathcal{L}_\rho)$ vanishes, since $C \cong \mathbb{P}_{3,1}$ by Observation 9.2. This implies that $R^\bullet \pi_* \mathcal{E} = \bigoplus_{\rho \in \mathbf{R}} R^\bullet \pi_* \mathcal{L}_\rho$ is a complex of vector bundles. A basic property of virtual fundamental classes ([5], Proposition 5.6) now states that $[F'_\beta]^{\text{vir}} = e((R^1 \pi_* \mathcal{E})^{\mathbb{C}^*})$.

By definition, $I^\theta(q, \hbar)$ is the contribution to the equivariant integral

$$\sum_{\beta_j} q^{\beta - \beta_0(\theta, 1)} \gamma_j \langle \gamma^j \text{ev}_1^*[\infty] \rangle_{1, \beta}^{0+, \theta, Gr}$$

coming from the loci F_β of Definition 6.5. By the projection formula, this is the contribution from the loci F'_β to:

$$(24) \quad \sum_{\beta, j} q^{\beta - \beta_0(\theta, 1)} \iota^*(\tilde{\gamma}_j) \int_{[\text{LGQG}_{0,1}^{0+}(X(\theta), \beta)]^{\text{vir}}} \text{ev}_1^*(\tilde{\gamma}^j \otimes [\infty]),$$

with $\iota^*(\tilde{\gamma}_j) = \gamma_j$, $\iota^*(\tilde{\gamma}^j) = \gamma^j$, and

$$\langle \underline{Z} \cup \tilde{\gamma}_j, \tilde{\gamma}^{j'} \rangle = \delta_j^{j'}.$$

We may choose isomorphisms of $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z$, and \mathcal{L}_a with the line bundles $\mathcal{O}_{\mathbb{P}_{3,1}}(\beta_x), \mathcal{O}_{\mathbb{P}_{3,1}}(\beta_y), \mathcal{O}_{\mathbb{P}_{3,1}}(\beta_z), \mathcal{O}_{\mathbb{P}_{3,1}}(\beta_a)$, and write σ as a tuple of sections

$$(\sigma_{x_0}(s, t), \sigma_{x_1}(s, t), \sigma_{x_2}(s, t), \sigma_{y_0}(s, t), \dots, \sigma_{p_y}(s, t), \sigma_{p_z}(s, t)),$$

where the entries are homogeneous polynomials in s and t of the appropriate degrees, and the degrees of s and t are 3 and 1, respectively. The fact that σ is \mathbb{C}^* -fixed implies that σ is of the form

$$(25) \quad \sigma = (x_0 s^{\beta_x - \beta_a}, x_1 s^{\beta_x}, x_2 s^{\beta_x}, y_0 s^{\beta_y - \beta_a}, \dots, p_y s^{-3\beta_y - 1}, p_z s^{-3\beta_z - 1}).$$

In particular, this shows

Proposition 9.4. Fix β so that $(\beta, 1)$ is θ -effective. The map $\text{ev}_1 : F'_\beta \rightarrow \bar{I}X(\theta)$ is an embedding.

Definition 9.5. On $\mathbb{P}_{3,1}$, a line bundle \mathcal{L} is determined up to isomorphism by its degree β ; the fractional part $\langle \beta \rangle$ determines the multiplicity $\text{mult}_\infty(\mathcal{L})$ at the orbifold point. In turn, $\text{mult}_\infty(\mathcal{L})$ determines a component of $\bar{I}X(\theta)$, which we denote $X(\theta)_{\langle \beta \rangle}$, such that ev_1 factors through $X(\theta)_{\langle \beta \rangle} \hookrightarrow \bar{I}X(\theta)$. We denote the fundamental class of this sector by $1_{\langle \beta \rangle}$.

Lemma 9.6. *The image of $\text{ev}_1 : F'_\beta \hookrightarrow X(\theta)_{\langle \beta \rangle}$ is the substack of $X_R(\theta)_{\langle \beta \rangle} := X(\theta)_{\langle \beta \rangle} \cap X_R(\theta)$ cut out by the vanishing of x_0 , if x is a superscript variable and $\beta_x - \beta_a \in \mathbb{Z}_{<0}$. (Similarly cut out by the vanishing of y_0 and z_0 .)*

Proof. An entry of σ in (25) is necessarily zero either of the following holds:

- (1) The corresponding line bundle \mathcal{L}_ρ has degree $\beta_\rho \notin \mathbb{Z}$. In this case s^{β_ρ} does not make sense.
- (2) \mathcal{L}_ρ has degree $\beta_\rho \in \mathbb{Z}_{<0}$. In this case \mathcal{L}_ρ has no nonzero global sections.

The first case imposes no restriction on $X(\theta)_{\langle \beta \rangle}$. In other words, if $\beta_\rho \notin \mathbb{Z}$, then the corresponding coordinate vanishes on $X(\theta)_{\langle \beta \rangle}$.

The second case does impose restrictions. First, if x is a superscript variable, we must have $\sigma_{p_x} = 0$. (In fact, we already observed that u factors through $X_R(\theta)$.) This shows that $\text{Im}(\text{ev}_1) \subseteq X_R(\theta)_{\langle \beta \rangle}$. Also, if $\beta_x - \beta_a \in \mathbb{Z}_{<0}$, then $\sigma_{x_0} = 0$. \square

Proposition 9.7. *If x is a subscript variable, only terms of $I^\theta(q, \hbar)$ with $\beta_x - \beta_0(\theta, 1) \in \frac{1}{3}\mathbb{Z}_{<0} \setminus \mathbb{Z}$ are nonzero. If x is a superscript variable, only terms of $I^\theta(q, \hbar)$ with $\beta_x \in \mathbb{Z}_{\geq 0}$ are nonzero.*

Proof. The first claim is immediate from $\beta_x - \beta_0(\theta, 1) < 0$ (Section 4.2) and Remark 5.13. For the second, Definition 4.1 implies that $\beta_x \geq 0$. If $\beta_x \notin \mathbb{Z}$, by the fact that σ does not vanish at $\infty \in \mathbb{P}_{3,1}$, we have $x_1 = x_2 = 0$ in (25). In particular, the sector $X_R(\theta)_{\langle \beta \rangle}$ is supported over the locus $x_1 = x_2 = 0$. Thus for any nonzero term $c\gamma_j q^\beta$ of $I^\theta(q, \hbar)$, where c is a scalar and $\beta_x \notin \mathbb{Z}$, we must have $\gamma_j \in \iota^*(H^*(X_R(\theta)_{\beta-1})) = 0$. \square

This calculation is adapted from [7].

Proposition 9.8. *The virtual normal bundle to F'_β inside $\text{LGQG}_{0,1}^{0+}(X(\theta), \beta)$ has \mathbb{C}^* -equivariant Euler class*

$$e^{\mathbb{C}^*}(N_{F'_\beta}^{\text{vir}}) = (-\hbar) \frac{\prod_{\substack{\rho \in \mathbf{R} \\ \beta_\rho \geq 0}} \prod_{0 \leq \nu \leq \lceil \beta_\rho \rceil - 1} (\text{ev}_1^* D_\rho + (\beta_\rho - \nu)\hbar)}{\prod_{\substack{\rho \in \mathbf{R} \\ \beta_\rho < 0}} \prod_{\lfloor \beta_\rho \rfloor + 1 \leq \nu \leq -1} (\text{ev}_1^* D_\rho + (\beta_\rho - \nu)\hbar)}.$$

where the divisors D_ρ were defined in Section 3.3.

Proof. The factor $(-\hbar)$ is from moving the marked point on C , and the rest comes from the relative perfect obstruction theory $R^\bullet \pi_* \mathcal{E}$. By Observation 9.3, each summand \mathcal{L}_ρ of \mathcal{E} contributes either $(\pi_* \mathcal{L})^{\text{mov}}$ or $-(R^1 \pi_* \mathcal{L})^{\text{mov}}$ to the virtual normal bundle, whichever is nonzero, where the superscript ‘mov’ denotes the \mathbb{C}^* -moving invariant subbundle.

We see from the form of (25) that for all $\rho \in \mathbf{R}$ with $\beta_\rho \in \mathbb{Z}_{\geq 0}$, we have

$$(26) \quad \mathcal{L}_\rho \cong \pi^* \text{ev}_1^* L_\rho \otimes \varpi^* \mathcal{O}_{\mathbb{P}_{3,1}}(3\beta_\rho).$$

(Recall that $\mathcal{L}_\rho = u^* L_\rho$ and that $\varpi : U'_\beta \rightarrow \mathbb{P}_{3,1}$ is the projection.)

Claim. Equation (26) holds, at least up to torsion, for all $\rho \in \mathbf{R}$.

Proof of Claim. We check separately for each $\theta \in \Theta$. For all θ , $\{a = 0\} \subseteq V^{\text{uns}}(\theta)$, so Definition 4.1 implies that $3\beta_a \in \mathbb{Z}_{\geq 0}$. Up to torsion, this identifies

$$(27) \quad \mathcal{L}_a \cong \pi^* \text{ev}_1^* L_{\hat{a}} \otimes \varpi^* \mathcal{O}_{\mathbb{P}_{3,1}}(3\beta_a).$$

For $\theta = \theta^{xyz a}$, at least one of $\beta_x - \beta_a$ and β_x is a nonnegative integer. By commutativity of tensor products and pullbacks, the fact that Equation (26) holds for one of \mathcal{L}_x and $\mathcal{L}_x \otimes \mathcal{L}_a^*$, together with

Equation (27), implies that Equation (26) holds for the other. A similar argument works for y and z .

It remains to check Equation (26) for $\rho = -3\hat{t}_x + \hat{t}_R$. Since $\mathcal{L}_R \cong \omega_{U'_\beta/F'_\beta, \log}$, the triviality of U'_β over F'_β implies $\mathcal{L}_{\hat{t}_R} \cong \varpi^* \mathcal{O}_{\mathbb{P}_{3,1}}(-3)$.

For $\theta = \theta_{yz}^a$, we must have $\beta_{-3\hat{t}_x + \hat{t}_R} = -3\beta_x + \beta_{\hat{t}_R} \in \mathbb{Z}_{\geq 0}$. Therefore Equation (26) implies

$$\begin{aligned} \mathcal{L}_{-3\hat{t}_x + \hat{t}_R} &\cong \pi^* \text{ev}_1^* L_{-3\hat{t}_x} \otimes \varpi^* \mathcal{O}_{\mathbb{P}_{3,1}}(-9\beta_x + 3\beta_{\hat{t}_R}) \\ &= \pi^* \text{ev}_1^* L_{-3\hat{t}_x} \otimes \varpi^* \mathcal{O}_{\mathbb{P}_{3,1}}(-9\beta_x - 3). \end{aligned}$$

Again we have $\mathcal{L}_{\hat{t}_R} \cong \varpi^* \mathcal{O}_{\mathbb{P}_{3,1}}(-3)$, so

$$\mathcal{L}_{-3\hat{t}_x} = \mathcal{L}_x^{\otimes -3} \cong \pi^* \text{ev}_1^* L_{-3\hat{t}_x} \otimes \varpi^* \mathcal{O}_{\mathbb{P}_{3,1}}(-9\beta_x).$$

Thus up to torsion this identifies $\mathcal{L}_x \cong \pi^* \text{ev}_1^* L_{\hat{t}_x} \otimes \varpi^* \mathcal{O}_{\mathbb{P}_{3,1}}(3\beta_x)$. The argument used for θ^{xyza} to describe \mathcal{L}_a applies here to show that Equation (26) holds for $\mathcal{L}_{\hat{t}_x - \hat{t}_a}$. The same argument works for the characters $\hat{t}_y - \hat{t}_a$, \hat{t}_y , $\hat{t}_z - \hat{t}_a$, and \hat{t}_z . These two arguments together prove the claim for θ_z^{xya} and θ_{yz}^{xa} also. \square

Now, by the projection formula,

$$\begin{aligned} R^i \pi_*(\mathcal{E}) &= \bigoplus_{\rho \in \mathbf{R}} R^i \pi_*(\pi^* \text{ev}_1^* L_\rho \otimes \varpi^* \mathcal{O}_{\mathbb{P}_{3,1}}(3\beta_\rho)) \\ &= \bigoplus_{\rho \in \mathbf{R}} \text{ev}_1^* L_\rho \otimes R^i \pi_* \varpi^* \mathcal{O}_{\mathbb{P}_{3,1}}(3\beta_\rho). \end{aligned}$$

Now $R^i \pi_* \varpi^* \mathcal{O}_{\mathbb{P}_{3,1}}(3\beta_\rho)$ is trivial with fiber $H^i(\mathbb{P}_{3,1}, \mathcal{O}_{\mathbb{P}_{3,1}}(3\beta_\rho))$, i.e.

$$R^i \pi_*(\mathcal{E}) = \bigoplus_{\rho \in \mathbf{R}} \text{ev}_1^* L_\rho \otimes H^i(\mathbb{P}_{3,1}, \mathcal{O}_{\mathbb{P}_{3,1}}(3\beta_\rho)).$$

The \mathbb{C}^* -action on $\text{LGQG}_{0,1}^{0+}(Z(\theta), \beta)$ is induced from an action on $\mathbb{P}_{3,1}$ via the universal map $\tau : \text{UQG}_{0,1}^{0+}(Z(\theta), \beta) \rightarrow \mathbb{P}_{3,1}$. Restricting to F'_β shows that the action on $R^i \pi_*(\mathcal{E})$ is induced by the projection $\varpi : U'_\beta \rightarrow \mathbb{P}_{3,1}$. Thus the \mathbb{C}^* -action on each factor $\text{ev}_1^* L_\rho \otimes H^i(\mathbb{P}_{3,1}, \mathcal{O}_{\mathbb{P}_{3,1}}(3\beta_\rho))$ is the natural action on $H^i(\mathbb{P}_{3,1}, \mathcal{O}_{\mathbb{P}_{3,1}}(3\beta_\rho))$.

As these groups are identified with the tangent and obstruction spaces at the point σ of Equation (25), the natural \mathbb{C}^* -action on a section $s^a t^b \in H^i(\mathbb{P}_{3,1}, \mathcal{O}_{\mathbb{P}_{3,1}}(3\beta_\rho))$ has weight $b/3$. The sections of $H^i(\mathbb{P}_{3,1}, \mathcal{O}_{\mathbb{P}_{3,1}}(3\beta_\rho))$ are, explicitly,

$$\begin{aligned} H^0(\mathbb{P}_{3,1}, \mathcal{O}_{\mathbb{P}_{3,1}}(3\beta_\rho)) &= \begin{cases} \mathbb{C}\{t^{3\beta_\rho}, st^{3(\beta_\rho-1)} \dots, s^{\lfloor \beta_\rho \rfloor} t^{3\langle \beta_\rho \rangle}\} & \beta_\rho \geq 0 \\ 0 & \beta_\rho < 0 \end{cases} \\ H^1(\mathbb{P}_{3,1}, \mathcal{O}_{\mathbb{P}_{3,1}}(3\beta_\rho)) &= \begin{cases} 0 & \beta_\rho \geq -1 \\ \mathbb{C}\{s^{-1}t^{3(\beta_\rho+1)}, s^{-2}t^{3(\beta_\rho+2)} \dots, s^{\lfloor \beta_\rho \rfloor + 1} t^{3(\langle \beta_\rho \rangle - 1)}\} & \beta_\rho < -1 \end{cases}. \end{aligned}$$

Recalling the notation D_ρ of Section 3.3, we have

$$(28) \quad e_{\mathbb{C}^*}(R^0 \pi_*(\mathcal{E})) = \prod_{\substack{\rho \in \mathbf{R} \\ \beta_\rho \geq 0}} \prod_{\substack{0 \leq \nu \leq \lfloor \beta_\rho \rfloor \\ \nu \in \mathbb{Z}}} (\text{ev}_1^* D_\rho + (\beta_\rho - \nu)\hbar)$$

$$e_{\mathbb{C}^*}(R^1 \pi_*(\mathcal{E})) = \prod_{\substack{\rho \in \mathbf{R} \\ \beta_\rho < -1}} \prod_{\substack{\lfloor \beta_\rho \rfloor + 1 \leq \nu \leq -1 \\ \nu \in \mathbb{Z}}} (\text{ev}_1^* D_\rho + (\beta_\rho - \nu)\hbar)$$

$$(29) \quad e_{\mathbb{C}^*}(N_{F'_\beta}^{\text{vir}}) = (-\hbar) \frac{\prod_{\substack{\rho \in \mathbf{R} \\ \beta_\rho \geq 0}} \prod_{0 \leq \nu \leq \lfloor \beta_\rho \rfloor - 1} (\text{ev}_1^* D_\rho + (\beta_\rho - \nu)\hbar)}{\prod_{\substack{\rho \in \mathbf{R} \\ \beta_\rho < -1}} \prod_{\lfloor \beta_\rho \rfloor + 1 \leq \nu \leq -1} (\text{ev}_1^* D_\rho + (\beta_\rho - \nu)\hbar)}$$

□

Observation 9.9. *The calculation above also shows that the obstruction bundle $R^1 \pi_* \mathcal{E}$ has no \mathbb{C}^* -fixed part, i.e. $[F'_\beta]^{\text{vir}} = [F'_\beta]$.*

Remark 9.10. For ρ such that $\beta_\rho \in \mathbb{Z}$, the section $s^{\lfloor \beta_\rho \rfloor} t^{3\langle \beta_\rho \rangle} = s^{\beta_\rho}$ is \mathbb{C}^* -fixed, and thus is not part of the virtual normal bundle. This explains the difference in indexing between (28) and (29). The missing terms span the tangent space to F'_β .

Remark 9.11. For a fixed θ , the conditions of Definition 4.1 (with $m = 1$) determine the signs of β_x , β_y , β_z , and β_a . This determines the signs of β_ρ for all $\rho \in \mathbf{R}$ except for $\rho \in \{\widehat{t}_x - \widehat{t}_a, \widehat{t}_y - \widehat{t}_a, \widehat{t}_z - \widehat{t}_a\}$. Specifically, if $\beta_x \geq 0$, the quantity $\beta_x - \beta_a$ changes sign depending on whether $\beta_x \geq \beta_a$ or $\beta_x < \beta_a$. (Similarly for y and z .) Therefore, in view of Proposition 9.8, we will have to treat the case $0 \leq \beta_x < \beta_a$ separately in what follows. (See Lemma 10.4.)

Proposition 9.12.

$$\iota_*^R(I^\theta(q, \hbar)) = \sum_{\beta} q^{\beta - \beta_0(\theta, 1)} \frac{\prod_{\substack{\rho \in \mathbf{R} \\ \beta_\rho < -1}} \prod_{\lfloor \beta_\rho \rfloor + 1 \leq \nu \leq -1} (D_\rho + (\beta_\rho - \nu)\hbar)}{\prod_{\substack{\rho \in \mathbf{R} \\ \beta_\rho \geq 0}} \prod_{0 \leq \nu \leq \lfloor \beta_\rho \rfloor - 1} (D_\rho + (\beta_\rho - \nu)\hbar)} A_x A_y A_z 1_{\langle -\beta \rangle},$$

where ι_*^R is the embedding $Z(\theta) \hookrightarrow X_R^{\text{rig}}(\theta)$,

$$A_x = \begin{cases} D_x - D_a & 0 \leq \beta_x < \beta_a, \quad \beta_x - \beta_a \in \mathbb{Z} \\ 1 & \text{otherwise,} \end{cases}$$

and similarly for A_y and A_z .

Proof. Write $e_{\mathbb{C}^*}(N_{F'_\beta}^{\text{vir}}) = (-\hbar) \text{ev}_1^* \alpha$ from (29). The projection formula gives

$$\iota_* \left(\frac{1}{e_{\mathbb{C}^*}(N_{F'_\beta}^{\text{vir}})} \cap [F'_\beta] \right) = \frac{1}{-\alpha \hbar} \cap (\text{ev}_1)_* [F'_\beta] \in H_*(\overline{IX}(\theta)),$$

so we have

$$\begin{aligned}
\underline{Z} \sum_{\beta,j} q^{\beta-\beta_0(\theta,1)} \tilde{\gamma}_j \int_{[F'_\beta]^{\text{vir}}} \frac{\text{ev}_1^*(\tilde{\gamma}^j \otimes [\infty])}{e^{\mathbb{C}^*}(N_{F'_\beta}^{\text{vir}})} &= \underline{Z} \sum_{\beta,j} q^{\beta-\beta_0(\theta,1)} \tilde{\gamma}_j \int_{(\text{ev}_1)_*[F'_\beta]} \frac{(-\hbar)\tilde{\gamma}^j}{-\alpha\hbar} \\
&= \sum_{\beta,j} q^{\beta-\beta_0(\theta,1)} (\underline{Z} \cup \tilde{\gamma}_j) \int_{[X_R^{\text{rig}}(\theta)]} \frac{\tilde{\gamma}^j}{\alpha} \cup (A_x A_y A_z 1_{\langle\beta\rangle}) \\
&= \sum_{\beta} q^{\beta-\beta_0(\theta,1)} v^* \left(\frac{A_x A_y A_z 1_{\langle\beta\rangle}}{\alpha} \right) \\
&= \sum_{\beta} q^{\beta-\beta_0(\theta,1)} \frac{A_x A_y A_z 1_{\langle-\beta\rangle}}{\alpha}.
\end{aligned}$$

The second equality follows from Lemma 9.6 and the last equality follows from the fact that α and $A_x A_y A_z$ are classes on the untwisted sector of $\bar{I}X(\theta)$. \square

Definition 9.13. Write

$$\iota_* I^\theta(q, \hbar) = \sum_{\beta} q^\beta I_\beta^\theta(\hbar).$$

The *small Givental I^θ -function* $I^{\theta, \text{Giv}}(q, \hbar)$ is defined to be

$$(30) \quad I^{\theta, \text{Giv}}(q, \hbar) = \sum_{\beta} q^{\beta + \frac{1}{\hbar}(H_x, H_y, H_z, H_a)} (-1)^{3\beta_x + 3\beta_y + 3\beta_z} I_\beta^\theta(\hbar).$$

By Section 3.3 we have $H_a = 0$, and $H_x = 0$ if x is a subscript variable. (Similarly for y and z .)

In particular, Proposition 9.12 gives

$$\begin{aligned}
I^{\theta, \text{Giv}}(q, \hbar) &= \sum_{\beta} q_x^{\beta_x + H_x/\hbar} q_y^{\beta_y + H_y/\hbar} q_z^{\beta_z + H_z/\hbar} q_a^{\beta_a} (-1)^{3\beta_x + 3\beta_y + 3\beta_z} \\
&\quad \cdot \left(\frac{\prod_{\substack{\rho \in \mathbf{R} \\ \beta_\rho < -1}} \prod_{\lfloor \beta_\rho \rfloor + 1 \leq \nu \leq -1} (D_\rho + (\beta_\rho - \nu)\hbar)}{\prod_{\substack{\rho \in \mathbf{R} \\ \beta_\rho \geq 0}} \prod_{0 \leq \nu \leq \lfloor \beta_\rho \rfloor - 1} (D_\rho + (\beta_\rho - \nu)\hbar)} \right) A_x A_y A_z 1_{\langle\beta\rangle}.
\end{aligned}$$

Remark 9.14. In [7], there is defined a *big I -function* $\mathbb{I}(t, q, \hbar)$, also on the Lagrangian cone, where t is restricted to the untwisted sectors of $\mathcal{H}(\theta)$. We may mimic their construction with no modification. $I^{\theta, \text{Giv}}(q, \hbar)$ is obtained from the result by restricting t further to only untwisted degree 2 classes, adding the factor $(-1)^{3\beta_x + 3\beta_y + 3\beta_z}$, and finally by identifying $q = e^t$.

This identification seems mysterious, especially as the symbol $q^{H_x/\hbar}$ is otherwise meaningless. In fact, the identification arises naturally from the divisor equation in Gromov-Witten theory, see Remark 3.1.2 of [10]. It is an important part of the formal analytic continuation of Section 10.2.

The choice of sign in (30) comes from [6], in which degree d Gromov-Witten invariants of a quintic threefold *with fields* (which play a similar role to the LG-quasimap invariants we use) differ from the usual Gromov-Witten invariants of the quintic threefold by the sign $(-1)^{5d+1}$.

10. RELATING THE GENERATING FUNCTIONS OF THE DIFFERENT QUOTIENTS

In this section, we use a similar method to that in [10] to relate the I -functions $I^{\theta, \text{Giv}}(q, \hbar)$ for various θ . The method involves analytic continuation of the Γ -function, and to make sense of this we set up some minor formalism.

10.1. The Γ -function on $\mathbb{C} \times \mathbb{C}$.

Definition 10.1. For $s + \xi \in \mathbb{C} \times \mathbb{C}$, we define the *extended Γ -function* $\tilde{\Gamma} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}[[\xi]]$ by

$$\tilde{\Gamma}(s + \xi) := \sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(s)}{k!} \xi^k.$$

Intuitively, we take ξ to be an extremely small complex number.

Observation 10.2. *It is an easy exercise that $\tilde{\Gamma}$ satisfies the functional equation $\tilde{\Gamma}(s + \xi) = (s - 1 + \xi)\tilde{\Gamma}(s - 1 + \xi)$. This agrees with the intuition that $s + \xi$ is a complex number “near” s .*

It is easy to extend this to $\mathbb{C} \times \mathbb{C}^n$, and we get a map $\tilde{\Gamma}$ to $\mathbb{C}[[\xi_1, \dots, \xi_n]]$. In the next section, we use the functional equation to rewrite $I^{\theta, \text{Giv}}(t, q, \hbar)$ in terms of $\tilde{\Gamma}$. This will allow us to carry out (formal) analytic continuation.

Notation 10.3. Hereafter we drop the tilde from $\tilde{\Gamma}$.

10.2. Analytic continuation. First, we note that the seeming inconsistency in Remark 9.11 can be conveniently ignored, as follows.

Lemma 10.4. *For $\beta_x \geq 0$, the factor in $I^{\theta, \text{Giv}}(t, q, \hbar)$ corresponding to $\rho_{x_0} = \hat{t}_x - \hat{t}_a$ is equal to*

$$\hbar^{-\lceil \beta_x - \beta_a \rceil} \frac{\Gamma(\text{ev}_1^* D_\rho / \hbar + \beta_x - \beta_a - \lceil \beta_x - \beta_a \rceil + 1)}{\Gamma(1 + \text{ev}_1^* D_\rho / \hbar + \beta_x - \beta_a)}.$$

In particular, this holds whether $\beta_x < \beta_a$ or $\beta_x \geq \beta_a$.

Proof. If $\beta_x \geq \beta_a$, we have $A_x = 1$, and the corresponding factor in $I^{\theta, \text{Giv}}(t, q, \hbar)$ is by definition

$$\frac{A_x}{\prod_{0 \leq \nu \leq \lceil \beta_x - \beta_a \rceil - 1} (\text{ev}_1^* D_\rho + (\beta_x - \beta_a - \nu)\hbar)} = \hbar^{-\lceil \beta_x - \beta_a \rceil} \frac{1}{\prod_{0 \leq \nu \leq \lceil \beta_x - \beta_a \rceil - 1} (\text{ev}_1^* D_\rho / \hbar + \beta_x - \beta_a - \nu)}.$$

Formally using $\prod_{0 \leq \nu \leq k} (\alpha - \nu) = \frac{\Gamma(1+\alpha)}{\Gamma(\alpha-k)}$, we can write this as

$$\hbar^{-\lceil \beta_x - \beta_a \rceil} \frac{1}{\frac{\Gamma(1 + \text{ev}_1^* D_\rho / \hbar + \beta_x - \beta_a)}{\Gamma(\text{ev}_1^* D_\rho / \hbar + \beta_x - \beta_a - \lceil \beta_x - \beta_a \rceil + 1)}} = \hbar^{-\lceil \beta_x - \beta_a \rceil} \frac{\Gamma(\text{ev}_1^* D_\rho / \hbar + \beta_x - \beta_a - \lceil \beta_x - \beta_a \rceil + 1)}{\Gamma(1 + \text{ev}_1^* D_\rho / \hbar + \beta_x - \beta_a)}.$$

Meanwhile, if $\beta_x < \beta_a$, we have $A_x = D_\rho$, and the corresponding factor in $I^{\theta, \text{Giv}}(t, q, \hbar)$ is

$$\begin{aligned} & \left(\prod_{\lceil \beta_x - \beta_a \rceil + 1 \leq \nu \leq -1} (\text{ev}_1^* D_\rho + (\beta_x - \beta_a - \nu)\hbar) \right) A_x \\ &= \prod_{\lceil \beta_x - \beta_a \rceil \leq \nu \leq -1} (\text{ev}_1^* D_\rho + (\beta_x - \beta_a - \nu)\hbar) \\ &= \hbar^{-\lceil \beta_x - \beta_a \rceil} \frac{\Gamma(1 + \text{ev}_1^* D_\rho / \hbar + \beta_x - \beta_a - \lceil \beta_x - \beta_a \rceil)}{\Gamma(\text{ev}_1^* D_\rho / \hbar + \beta_x - \beta_a + 1)}. \end{aligned}$$

□

Because of this observation, we no longer need to treat the case $0 \leq \beta_x < \beta_a$ separately (and similarly for y and z). We now introduce notation making use of this. Let:

$$I^x(\beta) = q_x^{\beta_x + H_x/\hbar} (-1)^{3\beta_x} \frac{\left(\frac{\Gamma(1-3H_x/\hbar-1)}{\Gamma(-3H_x/\hbar-3\beta_x)} \right)}{\left(\frac{\Gamma(1+H_x/\hbar+\beta_x-\beta_a)}{\Gamma(H_x/\hbar-\langle\beta_a\rangle+1)} \right) \left(\frac{\Gamma(1+H_x/\hbar+\beta_x)}{\Gamma(H_x/\hbar+1)} \right)^2}$$

$$I_x(\beta) = q_x^{\beta_x} (-1)^{3\beta_x} \frac{\left(\frac{\Gamma(\langle\beta_x-\beta_a\rangle)}{\Gamma(\beta_x-\beta_a+1)} \right) \left(\frac{\Gamma(\langle\beta_x\rangle)}{\Gamma(\beta_x+1)} \right)^2}{\left(\frac{\Gamma(-3\beta_x)}{\Gamma(1)} \right)}.$$

We similarly define $I^y(\beta), I_y(\beta), I^z(\beta), I_z(\beta)$. Finally we let

$$I^a(\beta) = q_a^{\beta_a} \left(\frac{1}{\Gamma(1+3\beta_a)} \right).$$

It is now straightforward to check (if one is very careful with indices of products) that:

$$I^{xyz a}(t, q, \hbar) = \sum_{\substack{\beta_x, \beta_y, \beta_z \in \mathbb{Z}_{\geq 0} \\ \beta_a \in \frac{1}{3}\mathbb{Z}_{\geq 0}}} I^x(\beta) I^y(\beta) I^z(\beta) I^a(\beta) 1_{\langle\beta\rangle}$$

$$I_z^{xy a}(t, q, \hbar) = \sum_{\substack{\beta_x, \beta_y \in \mathbb{Z}_{\geq 0} \\ \beta_a \in \frac{1}{3}\mathbb{Z}_{\geq 0} \\ \beta_z \in \frac{1}{3}\mathbb{Z}_{<0} \setminus \mathbb{Z}_{<0}}} I^x(\beta) I^y(\beta) I_z(\beta) I^a(\beta) 1_{\langle\beta\rangle}$$

$$I_{yz}^{xa}(t, q, \hbar) = \sum_{\substack{\beta_x \in \mathbb{Z}_{\geq 0} \\ \beta_a \in \frac{1}{3}\mathbb{Z}_{\geq 0} \\ \beta_y, \beta_z \in \frac{1}{3}\mathbb{Z}_{<0} \setminus \mathbb{Z}_{<0}}} I^x(\beta) I_y(\beta) I_z(\beta) I^a(\beta) 1_{\langle\beta\rangle}$$

$$I_{xyz}^a(t, q, \hbar) = \sum_{\substack{\beta_a \in \frac{1}{3}\mathbb{Z}_{\geq 0} \\ \beta_x, \beta_y, \beta_z \in \frac{1}{3}\mathbb{Z}_{<0} \setminus \mathbb{Z}_{<0}}} I_x(\beta) I_y(\beta) I_z(\beta) I^a(\beta) 1_{\langle\beta\rangle}$$

Essentially all we have done is to rewrite these functions formally using the identity

$$\prod_{0 \leq \nu \leq k} (\alpha - \nu) = \frac{\Gamma(1 + \alpha)}{\Gamma(\alpha - k)},$$

and using Lemma 10.4 in exceptional cases.

Lemma 10.5. *Fix β_y, β_z , and β_a . Then*

$$\sum_{\beta_x \in \mathbb{Z}_{\geq 0}} I^x(\beta) 1_{\langle\beta\rangle}$$

analytically continues to

$$\sum_{\substack{\beta_x = \frac{1}{3}\mathbb{Z}_{<0} \\ \beta_x \notin \mathbb{Z} \\ \beta_x - \beta_a \notin \mathbb{Z}}} \frac{-2\pi i}{3(e^{2\pi i(\beta_x - H_x/\hbar)} - 1)} \frac{\Gamma(H_x/\hbar - \langle\beta_a\rangle + 1) \Gamma(H_x/\hbar + 1)^2}{\Gamma(\langle\beta_x - \beta_a\rangle) \Gamma(\langle\beta_x\rangle)^2 \Gamma(3H_x/\hbar + 1)} I_x(\beta) 1_{\langle\beta\rangle}$$

Proof. We have

$$\sum_{\beta_x \in \mathbb{Z}_{\geq 0}} I^x(\beta) 1_{\langle \beta \rangle} = \sum_{\beta_x \in \mathbb{Z}_{\geq 0}} q_x^{\beta_x + H_x/\hbar} (-1)^{3\beta_x} \frac{\left(\frac{\Gamma(1-3H_x/\hbar-1)}{\Gamma(-3H_x/\hbar-3\beta_x)} \right)}{\left(\frac{\Gamma(1+H_x/\hbar+\beta_x-\beta_a)}{\Gamma(H_x/\hbar-\langle \beta_a \rangle+1)} \right) \left(\frac{\Gamma(1+H_x/\hbar+\beta_x)}{\Gamma(H_x/\hbar+1)} \right)^2} 1_{\langle \beta \rangle}$$

Using that fact that

$$\frac{\Gamma(1+\alpha)}{\Gamma(\alpha-k)} = (-1)^{k+1} \frac{\Gamma(1-\alpha+k)}{\Gamma(-\alpha)},$$

which can be easily deduced from the function equation, we can write

$$(-1)^{3\beta_x} \left(\frac{\Gamma(1-3H_x/\hbar-1)}{\Gamma(-3H_x/\hbar-3\beta_x)} \right) = \frac{\Gamma(1+3H_x/\hbar+3\beta_x)}{\Gamma(3H_x/\hbar+1)}$$

Now we have

$$\sum_{\beta_x \in \mathbb{Z}_{\geq 0}} I^x(\beta) 1_{\langle \beta \rangle} = \sum_{\beta_x \in \mathbb{Z}_{\geq 0}} q_x^{\beta_x + H_x/\hbar} \frac{\Gamma(1+3H_x/\hbar+3\beta_x)}{\Gamma(1+H_x/\hbar+\beta_x-\beta_a)\Gamma(1+H_x/\hbar+\beta_x)^2} \Phi(\langle \beta_a \rangle) 1_{\langle \beta \rangle},$$

where

$$\Phi(\langle \beta_a \rangle) = \frac{\Gamma(H_x/\hbar - \langle \beta_a \rangle + 1) \Gamma(H_x/\hbar + 1)^2}{\Gamma(3H_x/\hbar + 1)}.$$

We rewrite this last expression using residues:

$$(31) \quad \sum_{\beta_x \geq 0} 2\pi i \operatorname{Res}_{s=\beta_x} \left(\frac{q^{s+H_x/\hbar}}{e^{2\pi i s} - 1} \frac{\Gamma(1+3H_x/\hbar+3s)}{\Gamma(1+H_x/\hbar+(s-\beta_a))\Gamma(1+H_x/\hbar+s)^2} \right) \Phi(\langle \beta_a \rangle) 1_{\langle \beta \rangle}.$$

The expression in the large parentheses may be thought of as having simple poles at $s \in \mathbb{Z}$ and at $s + H_x/\hbar \in \frac{1}{3}\mathbb{Z}_{<0}$. (These loci are unions copies of \mathbb{C} inside \mathbb{C}^2 . The sum of residues can therefore be written as the integral

$$\left(\int_{-i\infty}^{i\infty} \frac{q^{s+H_x/\hbar}}{e^{2\pi i s} - 1} \frac{\Gamma(1+3H_x/\hbar+3s)}{\Gamma(1+H_x/\hbar+(s-\beta_a))\Gamma(1+H_x/\hbar+s)^2} ds \right) \Phi(\langle \beta_a \rangle) 1_{\langle \beta \rangle},$$

along a contour in \mathbb{C}^2 such that the poles $s \in \mathbb{Z}_{<0}$ and $s + H_x/\hbar \in \frac{1}{3}\mathbb{Z}_{<0}$ are on one side, and the poles $s \in \mathbb{Z}_{\geq 0}$ are on the other side. For simplicity, the contour may be chosen inside a slice $\mathbb{C} \times \{H'\}$, i.e. we may work with a contour integral in \mathbb{C} .

Integrals of this form have been well-studied for a long time, see page 49 of [4]. From there we see that the integral converges to (31) when $q \notin \mathbb{R}_{<0}$ and $|q| < 3^3$. When $q \notin \mathbb{R}_{<0}$ and $|q| > 3^3$, the integral converges to the sum over the remaining poles⁷.

First we consider the poles $s \in \mathbb{Z}_{<0}$. Consider the expression

$$(32) \quad \frac{\Gamma(1+3H_x/\hbar+3s)}{\Gamma(1+H_x/\hbar+s-\beta_a)\Gamma(1+H_x/\hbar+s)^2}$$

as a function of H_x/\hbar , treating H_x/\hbar as a (small) complex number, and s as a fixed negative integer. At $H_x/\hbar = 0$, the expression (32)

- has a zero of order 1 if $\langle \beta_a \rangle \neq 0$ (a pole from the numerator and two zeroes from the denominator),
- has a zero of order 1 if $\beta_a \geq s$, and
- has a zero of order 2 if $\langle \beta_a \rangle = 0$ and $\beta_a < s$.

⁷Lemma 3.3 of [24] instead puts 3^{-3} as the boundary, and [10] agrees.

In the first and second cases, H_x restricts to zero on F_β , see Lemma 9.6. In the last case, we know that $H_x^2 = 0$, since H_x is the hyperplane class on the 1-dimensional space E . Together, these say that the residues at the poles $s \in \mathbb{Z}_{<0}$ vanish when multiplied by $1_{\langle\beta\rangle}$.

Thus the analytic continuation is the sum

$$(33) \quad \sum_{\substack{\tilde{\beta}_x \in \frac{H_x}{h} + \frac{1}{3}\mathbb{Z}_{<0}}} 2\pi i \operatorname{Res}_{s=\tilde{\beta}_x} \left(\frac{q^{s+H_x/h}}{e^{2\pi i s} - 1} \frac{\Gamma(1 + 3H_x/h + 3s)}{\Gamma(1 + H_x/h + (s - \beta_a))\Gamma(1 + H_x/h + s)^2} \right) \Phi(\langle\beta_a\rangle) 1_{\langle\beta^{\text{old}}\rangle} \\ = \sum_{\substack{\beta_x \in \frac{1}{3}\mathbb{Z}_{<0}}} 2\pi i \operatorname{Res}_{s=\beta_x} \left(\frac{q^s}{e^{2\pi i(s-H_x/h)} - 1} \frac{\Gamma(1 + 3s)}{\Gamma(1 + (s - \beta_a))\Gamma(1 + s)^2} \right) \Phi(\langle\beta_a\rangle) 1_{\langle\beta^{\text{old}}\rangle}.$$

Here $e^{2\pi i(s-H_x/h)}$ should be interpreted as in Section 10.1, via its expansion at $H_x/h = 0$.

Remark 10.6. We write β^{old} rather than β to emphasize that it has not changed and in particular is independent of $\langle\beta_x\rangle$ (as it has been all along — β_x has been an integer). Later in this section we will use $1_{\langle\beta\rangle}$ to refer to an element of $\mathcal{H}(\theta')$, where θ' is obtained by changing x from a superscript variable to a subscript variable.

What remains is to calculate the residues. When $\beta_x \in \mathbb{Z}$ or $\beta_x - \beta_a \in \mathbb{Z}$, the residue in (33) vanishes because the simple pole in $\Gamma(1 + 3s)$ is canceled by the poles in $\Gamma(1 + (s - \beta_a))$, or $\Gamma(1 + s)$. The residue of $\Gamma(1 + 3s)$ at $\beta_x \in \frac{1}{3}\mathbb{Z}_{<0}$ is

$$\frac{(-1)^{3\beta_x+1}}{3\Gamma(-3\beta_x)}.$$

Thus we rewrite (33) as

$$\sum_{\substack{\beta_x \in \frac{1}{3}\mathbb{Z}_{<0} \\ \beta_x \notin \mathbb{Z} \\ \beta_x - \beta_a \notin \mathbb{Z}}} 2\pi i \left(\frac{q^{\beta_x}}{e^{2\pi i(\beta_x - H_x/h)} - 1} \frac{\left(\frac{(-1)^{3\beta_x+1}}{3\Gamma(-3\beta_x)} \right)}{\Gamma(1 + (\beta_x - \beta_a))\Gamma(1 + \beta_x)^2} \right) \Phi(\langle\beta_a\rangle) 1_{\langle\beta^{\text{old}}\rangle}.$$

Rearranging slightly gives

$$(34) \quad \sum_{\substack{\beta_x \in \frac{1}{3}\mathbb{Z}_{<0} \\ \beta_x \notin \mathbb{Z} \\ \beta_x - \beta_a \notin \mathbb{Z}}} \frac{2\pi i q^{\beta_x} (-1)^{3\beta_x+1}}{3(e^{2\pi i(\beta_x - H_x/h)} - 1)} \frac{\frac{\Gamma(\langle\beta_x - \beta_a\rangle)}{\Gamma(\beta_x - \beta_a + 1)} \left(\frac{\Gamma(\langle\beta_x\rangle)}{\Gamma(\beta_x + 1)} \right)^2}{\left(\frac{\Gamma(-3\beta_x)}{\Gamma(1)} \right) \Gamma(\langle\beta_x - \beta_a\rangle) \Gamma(\langle\beta_x\rangle)^2} \Phi(\langle\beta_a\rangle) 1_{\langle\beta^{\text{old}}\rangle} \\ = \sum_{\substack{\beta_x \in \frac{1}{3}\mathbb{Z}_{<0} \\ \beta_x \notin \mathbb{Z} \\ \beta_x - \beta_a \notin \mathbb{Z}}} \frac{-2\pi i}{3(e^{2\pi i(\beta_x - H_x/h)} - 1)} \frac{\Phi(\langle\beta_a\rangle)}{\Gamma(\langle\beta_x - \beta_a\rangle) \Gamma(\langle\beta_x\rangle)^2} I_x(\beta) 1_{\langle\beta^{\text{old}}\rangle}.$$

□

We now use Section 10.1 to expand the factors. That is, again using $H_x^2 = 0$, we have

$$\frac{1}{e^{2\pi i(\beta_x - H_x/h)} - 1} = \frac{1}{e^{2\pi i\beta_x} - 1} + \frac{2\pi i e^{2\pi i\beta_x}}{(e^{2\pi i\beta_x} - 1)^2} H_x/h$$

and

$$\Phi(\langle\beta_a\rangle) = \Gamma(1 - \langle\beta_a\rangle) + \Gamma(1 - \langle\beta_a\rangle)(\mathfrak{h}_{-\langle\beta_a\rangle}) H_x/h,$$

where $\mathfrak{h}_0 = 0$, $\mathfrak{h}_{-1/3} = \frac{\pi}{2\sqrt{3}} - \frac{3\log 3}{2}$, and $\mathfrak{h}_{-1/3} = -\frac{\pi}{2\sqrt{3}} - \frac{3\log 3}{2}$. Finally we write (34) as:

$$\sum_{\substack{\beta_x \in \frac{1}{3}\mathbb{Z}_{<0} \\ \beta_x \notin \mathbb{Z} \\ \beta_x - \beta_a \notin \mathbb{Z}}} \frac{-2\pi i}{3\Gamma(\langle\beta_x - \beta_a\rangle)\Gamma(\langle\beta_x\rangle)^2} \cdot \left(\frac{\Gamma(1 - \langle\beta_a\rangle)}{e^{2\pi i\beta_x} - 1} + \left(\frac{2\pi i e^{2\pi i\beta_x} \Gamma(1 - \langle\beta_a\rangle)}{(e^{2\pi i\beta_x} - 1)^2} + \frac{\Gamma(1 - \langle\beta_a\rangle)(\mathfrak{h}_{-\langle\beta_a\rangle})}{e^{2\pi i\beta_x} - 1} \right) \frac{H_x}{\hbar} \right) I_x(\beta) 1_{\langle\beta^{\text{old}}\rangle}.$$

What remains is to make the identification in Section 5.1. Namely, write $1_{\langle\beta^{\text{old}}\rangle} = (1 \otimes \alpha_y \otimes \gamma_z)_g \in \mathcal{H}(\theta)$, and define

$$\begin{aligned} 1_{\langle\beta^{\text{old}}\rangle} &\mapsto 1_{\zeta, \langle\beta^{\text{old}}\rangle} := (1_\zeta \otimes \alpha_y \otimes \gamma_z)_g \in \mathcal{H}(\theta') \\ H_x 1_{\langle\beta^{\text{old}}\rangle} &\mapsto 1_{\zeta^2, \langle\beta^{\text{old}}\rangle} := (1_{\zeta^2} \otimes \alpha_y \otimes \gamma_z)_g \in \mathcal{H}(\theta'). \end{aligned}$$

Thus the coefficients

$$\frac{-2\pi i}{3\Gamma(\langle\beta_x - \beta_a\rangle)\Gamma(\langle\beta_x\rangle)^2} \cdot \frac{\Gamma(1 - \langle\beta_a\rangle)}{e^{2\pi i\beta_x} - 1}$$

and

$$\frac{-2\pi i}{3\Gamma(\langle\beta_x - \beta_a\rangle)\Gamma(\langle\beta_x\rangle)^2} \cdot \left(\frac{2\pi i e^{2\pi i\beta_x} \Gamma(1 - \langle\beta_a\rangle)}{(e^{2\pi i\beta_x} - 1)^2} + \frac{\Gamma(1 - \langle\beta_a\rangle)(\mathfrak{h}_{-\langle\beta_a\rangle})}{e^{2\pi i\beta_x} - 1} \right)$$

define an isomorphism $\mathcal{H}(\theta) \rightarrow \mathcal{H}(\theta')$. We have proved:

Theorem 10.7 (LG/CY correspondence). *This isomorphism identifies the analytically continued I-function $I^{\theta, \text{Giv}}(q, \hbar)$ with $I^{\theta', \text{Giv}}(q, \hbar)$.*

11. NOTATION TABLE

Notation	Description
1_θ	Generator of $\mathcal{H}^0(\theta)$
1_g	The fundamental class of g twisted sectors of X
$1_{\langle\beta\rangle}$	Fundamental class of certain twisted sectors, Section 9
a	Coordinates on V
b_i	Marked point on C
BG	Stack of principal G -bundles, i.e. $BG = [\text{Spec } \mathbb{C}/G]$
B_0, B_∞	Sets of marked points of G-graph quasimap over $0, \infty \in \mathbb{P}^1$
β	Shorthand for $(\beta_x, \beta_y, \beta_z, \beta_a)$
$\beta(P)$	Degree of the basepoint of σ at P , Section 4
β^0, β^∞	Degrees of LG-graph quasimap supported over $0, \infty \in \mathbb{P}^1$
$\beta_0(\theta, m)$	Extremal degree of m -marked LG-quasimaps to $X(\theta)$
β_ρ	Degree of \mathcal{L}_ρ
$\beta_x, \beta_y, \beta_z, \beta_a, \beta_R$	Degrees of $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z, \mathcal{L}_a, \mathcal{L}_R$
$\bullet, \check{\bullet}$	Points of \widehat{C} mapping to $0, \infty \in \mathbb{P}^1$
\widehat{C}	m -marked genus zero twisted curve
C_0, C_∞	Components of a graph quasimap over $0, \infty \in \mathbb{P}^1$
U'_β	Universal curve over F'_β
\mathbb{C}_R^*	Group acting on V
\mathbb{C}_ρ	The representation associated to a character ρ
$\text{Crit}(W)$	Critical locus of W in V
\widehat{C}	Parametrized component of a graph quasimap

d_P	The order of a point P on C
D_ρ	Toric divisor on $X(\theta)$
ev_i	Evaluation maps to $Z(\theta)$ or $X(\theta)$
E	Elliptic curve in \mathbb{P}^2
\mathcal{E}	$\mathcal{P} \times_{(G \times \mathbb{C}_R^*)} V$
ϵ	Stability parameter, in $\mathbb{Q}_{>0}$
F_β, F'_β	Special components of \mathbb{C}^* -fixed LG-graph quasimaps to $Z(\theta), X(\theta)$
$F_{B_0, \beta^0}^{B_\infty, \beta^\infty}$	\mathbb{C}^* -fixed LG-graph quasimaps inducing partitions $B_0 \sqcup B_\infty$ and $\beta^0 + \beta^\infty$
G	$(\mathbb{C}^*)^4$
$\{\gamma_j\}, \{\gamma^j\}$	Basis and dual basis for $\mathcal{H}(\theta)$
H	Class $[L_0] = [L'] \in H^2(\mathbb{P}^2/\mu_3)$
H_x, H_y, H_z	Divisor classes on $[E^3/\mu_3]$
$\mathcal{H}(\theta)$	Compact type state space associated to θ
\hbar	Generator of \mathbb{C}^* -equivariant cohomology ring of a point
$I\mathcal{X}$	(Nonrigidified) inertia stack of \mathcal{X}
$\bar{I}\mathcal{X}$	Rigidified inertia stack of \mathcal{X}
$IX^{\text{nar}}, \bar{I}X^{\text{nar}}$	Narrow components of $IX, \bar{I}X$
$I^\theta(q, \hbar)$	I -function
ι	Embedding $Z \hookrightarrow X$ or $Z(\theta) \hookrightarrow X(\theta)$
κ	Isomorphism $\mathcal{L}_R \rightarrow \omega_{C, \log}$
$\ell^\sigma(P)$	Length of σ at P
L_0, L'	Certain lines in $[\mathbb{P}^2/\mu_3]$
L_ρ	Line bundle on $X(\theta)$ (resp. $[X(\theta)/\mathbb{C}_R^*]$) corresponding to character ρ of G (resp. $G \times \mathbb{C}_R^*$)
$\text{LGQ}_{0,m}^\epsilon(X(\theta), \beta),$	Stack of ϵ -stable genus zero m -marked LG-quasimaps to $X(\theta)$ (resp,
$\text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)$	$Z(\theta))$ of degree β
$\text{LGQ}_{0,m}^\epsilon(X(\theta), \beta),$	Stack of LG-graph quasimaps to $X(\theta)$ (resp. $Z(\theta))$
$\text{LGQ}_{0,m}^\epsilon(Z(\theta), \beta)$	
\mathcal{L}_ρ	$u^*(L_\rho)$
$\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z, \mathcal{L}_a, \mathcal{L}_R$	Line bundles on C built from \mathcal{P}
$\text{mult}_P(\mathcal{L})$	The multiplicity (monodromy) of \mathcal{L} at P
m	Number of marked points on C
μ, ν	T -fixed points of $\bar{I}X(\theta)$
μ_d	The group of d th roots of unity in \mathbb{C}^*
N^{vir}	Virtual normal bundle
p_x, p_y, p_z	Coordinates on V
P	Class $[P_0] = [P'] \in H^4(\mathbb{P}^2/\mu_3)$
P_0, P'	Certain points in $[\mathbb{P}^2/\mu_3]$
$\mathbb{P}_{3,1}^1$	\mathbb{P}^1 with an order 3 orbifold point at $[\infty]$
\mathcal{P}	Principal $G \times \mathbb{C}_R^*$ -bundle
π	Map from universal curve to moduli stack
q, q_x, q_y, q_z, q_a	Formal parameters keeping track of $\beta, \beta_x, \beta_y, \beta_z, \beta_a$
\mathbf{R}	$\{\rho_{x_0}, \dots, \rho_{p_z}\}$
$\rho_{x_0}, \dots, \rho_{p_z}$	Characters of $G \times \mathbb{C}_R^*$, which define V as a direct sum
s	Complexification of β_x
s, t	Coordinates on $\mathbb{P}_{3,1}^1$
σ	Section of \mathcal{E}
$\sigma_{x_0}, \dots, \sigma_{p_z}$	Components of σ , sections of \mathcal{L}_ρ for $\rho \in \mathbf{R}$

(t_x, t_y, t_z, t_a)	Element of G
$\widehat{t}_x, \widehat{t}_y, \widehat{t}_z, \widehat{t}_a, \widehat{t}_R$	Characters $(t_x, t_y, t_z, t_a, t_R) \mapsto t_x$, etc., of $G \times \mathbb{C}_R^*$
t	Coordinates of cohomology ring
t_R	Element of \mathbb{C}_R^*
T	$(\mathbb{C}^*)^{13}$
τ	Parametrization map $C \rightarrow \mathbb{P}^1$
$\theta, \theta^{xyza}, \dots, \theta_{xyz}^a$	GIT characters of G
$\theta, \theta^{xyza}, \dots, \theta_{xyz}^a$	Lifts of $\theta, \theta^{xyza}, \dots, \theta_{xyz}^a$ to $G \times \mathbb{C}_R^*$
Θ	$\{\theta^{xyza}, \theta_z^{xya}, \theta_{yz}^{xa}, \theta_{xyz}^a\}$
u	Map $C \rightarrow [V/(G \times \mathbb{C}_R^*)]$
V	\mathbb{C}^{13}
$V^{ss}(\theta)$	θ -semistable locus of V
$V^{uns}(\theta)$	θ -unstable locus of V
$[V //_{\theta} G]$	The GIT stack quotient $[V^{ss}(\theta)/G]$
$w(\mu, \nu)$	Tangent weight at μ along curve from μ to ν
W	Function $p_x(ax_0^3 + x_1^3 + x_2^3) + p_y(ay_0^3 + y_1^3 + y_2^3) + p_z(az_0^3 + z_1^3 + z_2^3)$
$\omega_{C, \log}$	Log canonical bundle of C
x_0, x_1, x_2	Coordinates on V
X	$[V/G]$
$X(\theta)$	$[V //_{\theta} G]$
$X(\theta)_{\langle \beta \rangle}$	Component of $\bar{I}X(\theta)$, Section 9
$X_R(\theta)$	Points $P \in [X(\theta)/\mathbb{C}_R^*]$ with $\mathbb{C}_R^* \subseteq G_P$
y_0, y_1, y_2	Coordinates on V
z_0, z_1, z_2	Coordinates on V
Z	$[\text{Crit}(W)/G]$
$Z(\theta)$	$[\text{Crit}(W) \cap V^{ss}(\theta)/G]$
ζ	$e^{2\pi i/3} \in \mu_3$

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